Indian Institute of Science
ME 261: Midsemester Test

Date: 29/9/18.
Duration: 9.30 a.m.–11.00 a.m.
Maximum Marks: 100

1. Find the factors $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ in the polar decomposition of (35)

$$\boldsymbol{F} = \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n},$$

where $\boldsymbol{n}$ is a unit vector. (Hint: $\boldsymbol{U}$ and $\boldsymbol{V}$ can be positive semi-definite, and $\boldsymbol{R}$ may not be unique; if it is not unique, then just present one $\boldsymbol{R}$ that works.). Next, find the principal invariants and eigenvalues of $\boldsymbol{F}$, $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{R}$.

2. Let $\boldsymbol{W} \in \text{Skw}$, and let $\boldsymbol{w}$ be its axial vector. Given that (30)

$$\boldsymbol{R} = e^{\boldsymbol{W}} = \boldsymbol{I} + \frac{\sin|\boldsymbol{w}|}{|\boldsymbol{w}|}\boldsymbol{W} + \frac{(1-\cos|\boldsymbol{w}|)}{|\boldsymbol{w}|^2}\boldsymbol{W}^2 \in \text{Orth}^+,$$

find an expression for $(e^{\boldsymbol{W}})^3$ of the form $\alpha_0(\boldsymbol{w})\boldsymbol{I} + \alpha_1(\boldsymbol{w})\boldsymbol{W} + \alpha_2(\boldsymbol{w})\boldsymbol{W}^2$ where you have to determine the $\alpha_i(\boldsymbol{w})$, $i = 0, 1, 2$. Next, determine if $(e^{\boldsymbol{W}})^3 \in \text{Orth}^+$. If it does, then find its axis. *Justify* all steps.

3. Let $\boldsymbol{x}$ denote the position vector, and let $\boldsymbol{u} = \boldsymbol{x}/|\boldsymbol{x}|^3$. Determine $\boldsymbol{\nabla} \cdot \boldsymbol{u}$, $\boldsymbol{\nabla} \times \boldsymbol{u}$ and $\boldsymbol{\nabla}^2\boldsymbol{u}$. (35) Determine the constant k if $\phi = k/|\boldsymbol{x}|$ and $\boldsymbol{\nabla}\phi = \boldsymbol{u}$. For an *arbitrary* $\boldsymbol{u}(\boldsymbol{x})$, find a relation between $\boldsymbol{\nabla}^2\boldsymbol{u}$, $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u})$ and $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u})$.

Some relevant formulae

$$\begin{aligned}
w_i &= -\frac{1}{2}\epsilon_{ijk}W_{jk}, \\
W_{ij} &= -\epsilon_{ijk}w_k, \\
(\mathbf{cof}\,\boldsymbol{T})_{ij} &= \frac{1}{2}\epsilon_{imn}\epsilon_{jpq}T_{mp}T_{nq}, \\
I_2(\boldsymbol{T}) &= \frac{1}{2}\left[(\text{tr}\,\boldsymbol{T})^2 - \text{tr}\,\boldsymbol{T}^2\right], \\
\det\boldsymbol{T} &= \epsilon_{ijk}T_{i1}T_{j2}T_{k3} = \epsilon_{ijk}T_{1i}T_{2j}T_{3k}
\end{aligned}$$