Engineering Maths

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Sets

Lin = Set of all tensors
Lin^+ = Set of all tensors $\boldsymbol{T}$ with $\det \boldsymbol{T} > 0$
Sym = Set of all symmetric tensors
Psym = Set of all symmetric, positive definite tensors
Orth = Set of all orthogonal tensors
Orth^+ = Set of all rotations ($\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}$ and $\det \boldsymbol{Q} = +1$)
Skw = Set of all skew-symmetric tensors

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Chapter 1

Introduction to Tensors

Throughout the text, scalars are denoted by lightface letters, vectors are denoted by boldface lower-case letters, while second and higher-order tensors are denoted by boldface capital letters. As a notational issue, summation over repeated indices is assumed, with the indices ranging from 1 to 3 (since we are assuming three-dimensional space). Thus, for example,

$$T_{ii} = T_{11} + T_{22} + T_{33},$$
$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i \overset{\text{OR}}{=} u_j v_j = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

The repeated indices i and j in the above equation are known as *dummy indices* since any letter can be used for the index that is repeated. Dummy indices can be repeated twice and twice only (i.e., not more than twice). In case, we want to denote summation over an index repeated three times, we use the summation sign *explicitly* (see, for example, Eqn. (1.98)). As another example, let $\boldsymbol{v} = \boldsymbol{T}\boldsymbol{u}$ denote a matrix $\boldsymbol{T}$ multiplying a vector $\boldsymbol{u}$ to give a vector $\boldsymbol{v}$. In terms of indicial notation, we write this equation as

$$v_i = T_{ij} u_j \overset{\text{OR}}{=} T_{ik} u_k. \tag{1.1}$$

Thus, by letting $i = 1$, we get

$$v_1 = T_{1j} u_j = T_{11} u_1 + T_{12} u_2 + T_{13} u_3.$$

We get the expressions for the other components of $\boldsymbol{v}$ by successively letting i to be 2 and then 3. In Eqn. (1.1), i denotes the free index, and j and k denote dummy indices. Note that the same number of free indices should occur on both sides of the equation. In the equation

$$T_{ij} = C_{ijkl} E_{kl},$$

i and j are free indices and k and l are dummy indices. Thus, we have

$$T_{11} = C_{11kl} E_{kl} = C_{1111} E_{11} + C_{1112} E_{12} + C_{1113} E_{13}$$

$$
\begin{aligned}
&+ C_{1121}E_{21} + C_{1122}E_{22} + C_{1123}E_{23}\\
&+ C_{1131}E_{31} + C_{1132}E_{32} + C_{1133}E_{33},\\
T_{12} = C_{12kl}E_{kl} &= C_{1211}E_{11} + C_{1212}E_{12} + C_{1213}E_{13}\\
&+ C_{1221}E_{21} + C_{1222}E_{22} + C_{1223}E_{23}\\
&+ C_{1231}E_{31} + C_{1232}E_{32} + C_{1233}E_{33}.
\end{aligned}
$$

The expressions for T_{13}, T_{21} etc. can be generated similar to the above.

The representation of $\boldsymbol{T} = \boldsymbol{RS}$ would be

$$T_{ij} = R_{ik}S_{kj} \overset{\text{OR}}{=} S_{kj}R_{ik}.$$

i and j are free indices and k is a dummy index. *Although we are allowed to interchange the order of R_{ik} and S_{kj} while writing the indicial form as shown in the equation (since they are scalars), we are not allowed to interchange the order while writing the tensorial form, i.e, we cannot write $\boldsymbol{T} = \boldsymbol{RS}$ as $\boldsymbol{T} = \boldsymbol{SR}$ since the two products $\boldsymbol{RS}$ and $\boldsymbol{SR}$ are different.*

Great care has to be exercised in using indicial notation. In particular, the rule of dummy indices not getting repeated more than twice should be strictly adhered to, as the following example shows. If $\boldsymbol{a} = \boldsymbol{Hu}$ and $\boldsymbol{b} = \boldsymbol{Gv}$, then we can write $a_i = H_{ij}u_j$ and $b_i = G_{ij}v_j$. But we cannot write

$$\boldsymbol{a} \cdot \boldsymbol{b} = H_{ij}u_jG_{ij}v_j, \quad \text{(wrong indicial representation!)}$$

since the index j is repeated more than twice. Thus, although $a_i = H_{ij}u_j$ and $b_i = G_{ij}v_j$ are both correct, one cannot blindly substitute them to generate $\boldsymbol{a} \cdot \boldsymbol{b}$. The correct way to write the above equation is

$$\boldsymbol{a} \cdot \boldsymbol{b} = H_{ij}u_jG_{ik}v_k \overset{\text{OR}}{=} H_{ij}G_{ik}u_jv_k \overset{\text{OR}}{=} u_jv_kH_{ij}G_{ik}, \quad \text{(correct indicial representation)}$$

where we have now introduced another dummy index k to prevent the dummy index j from being repeated more than twice. Write out the wrong and the correct expressions explicitly by carrying out the summation over the dummy indices to convince yourself about this result.

In what follows, the quantity on the right-hand side of a ':=' symbol defines the quantity on its left-hand side.

1.1 Vectors in $\Re^3$

From now on, V denotes the three-dimensional Euclidean space $\Re^3$. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be a fixed set of orthonormal vectors that constitute the Cartesian basis. We have

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij},$$

where δ_{ij}, known as the Kronecker delta, is defined by

$$\delta_{ij} := \begin{cases} 0 & \text{when } i \neq j, \\ 1 & \text{when } i = j. \end{cases} \tag{1.2}$$

The Kronecker delta is also known as the substitution operator, since, from the definition, we can see that $x_i = \delta_{ij} x_j$, $\tau_{ij} = \tau_{ik}\delta_{kj}$, and so on. Note that $\delta_{ij} = \delta_{ji}$, and $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$.

Any vector $\boldsymbol{u}$ can be written as

$$\boldsymbol{u} = u_1 \boldsymbol{e}_1 + u_2 \boldsymbol{e}_2 + u_3 \boldsymbol{e}_3, \tag{1.3}$$

or, using the summation convention, as

$$\boldsymbol{u} = u_i \boldsymbol{e}_i.$$

The inner product of two vectors is given by

$$(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v} := u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{1.4}$$

Using Eqn. (1.3), the components of the vector $\boldsymbol{u}$ can be written as

$$u_i = \boldsymbol{u} \cdot \boldsymbol{e}_i. \tag{1.5}$$

Substituting Eqn. (1.5) into Eqn. (1.3), we have

$$\boldsymbol{u} = (\boldsymbol{u} \cdot \boldsymbol{e}_i)\boldsymbol{e}_i. \tag{1.6}$$

We define the cross product of two base vectors $\boldsymbol{e}_j$ and $\boldsymbol{e}_k$ by

$$\boldsymbol{e}_j \times \boldsymbol{e}_k := \epsilon_{ijk} \boldsymbol{e}_i, \tag{1.7}$$

where ϵ_{ijk} is given by

$$\begin{aligned} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \\ \epsilon_{ijk} = 0 \quad \text{otherwise.} \end{aligned}$$

Taking the dot product of both sides of Eqn. (1.7) with $\boldsymbol{e}_m$, we get

$$\boldsymbol{e}_m \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) = \epsilon_{ijk}\delta_{im} = \epsilon_{mjk}.$$

Using the index i in place of m, we have

$$\epsilon_{ijk} = \boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k). \tag{1.8}$$

The cross product of two vectors is assumed to be distributive, i.e.,

$$(\alpha \boldsymbol{u}) \times (\beta \boldsymbol{v}) = \alpha\beta(\boldsymbol{u} \times \boldsymbol{v}) \quad \forall \alpha, \beta \in \Re \text{ and } \boldsymbol{u}, \boldsymbol{v} \in V.$$

If $\boldsymbol{w}$ denotes the cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$, then by using this property and Eqn. (1.7), we have

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{u} \times \boldsymbol{v} \\ &= (u_j \boldsymbol{e}_j) \times (v_k \boldsymbol{e}_k) \\ &= \epsilon_{ijk} u_j v_k \boldsymbol{e}_i. \end{aligned} \tag{1.9}$$

It is clear from Eqn. (1.9) that

$$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}.$$

Taking $\boldsymbol{v} = \boldsymbol{u}$, we get $\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0}$.

The scalar triple product of three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, denoted by $[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$, is defined by

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] := \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}).$$

In indicial notation, we have

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \epsilon_{ijk} u_i v_j w_k. \tag{1.10}$$

From Eqn. 1.10, it is clear that

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = [\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}] = [\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}] = -[\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}] = -[\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}] = -[\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}] \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V. \tag{1.11}$$

If any two elements in the scalar triple product are the same, then its value is zero, as can be seen by interchanging the identical elements, and using the above formula. From Eqn. (1.10), it is also clear that the scalar triple product is linear in each of its argument variables, so that, for example,

$$[\alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{x}, \boldsymbol{y}] = \alpha [\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y}] + \beta [\boldsymbol{v}, \boldsymbol{x}, \boldsymbol{y}] \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}, \boldsymbol{y} \in V. \tag{1.12}$$

As can be easily verified, Eqn. (1.10) can be written in determinant form as

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}. \tag{1.13}$$

Using Eqns. (1.8) and (1.13), the components of the alternate tensor can be written in determinant form as follows:

$$\epsilon_{ijk} = [\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k] = \det \begin{bmatrix} \boldsymbol{e}_i \cdot \boldsymbol{e}_1 & \boldsymbol{e}_i \cdot \boldsymbol{e}_2 & \boldsymbol{e}_i \cdot \boldsymbol{e}_3 \\ \boldsymbol{e}_j \cdot \boldsymbol{e}_1 & \boldsymbol{e}_j \cdot \boldsymbol{e}_2 & \boldsymbol{e}_j \cdot \boldsymbol{e}_3 \\ \boldsymbol{e}_k \cdot \boldsymbol{e}_1 & \boldsymbol{e}_k \cdot \boldsymbol{e}_2 & \boldsymbol{e}_k \cdot \boldsymbol{e}_3 \end{bmatrix} = \det \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}. \tag{1.14}$$

Thus, we have

$$
\begin{aligned}
\epsilon_{ijk}\epsilon_{pqr} &= \det\begin{bmatrix}\delta_{i1} & \delta_{i2} & \delta_{i3}\\ \delta_{j1} & \delta_{j2} & \delta_{j3}\\ \delta_{k1} & \delta_{k2} & \delta_{k3}\end{bmatrix} \det\begin{bmatrix}\delta_{p1} & \delta_{p2} & \delta_{p3}\\ \delta_{q1} & \delta_{q2} & \delta_{q3}\\ \delta_{r1} & \delta_{r2} & \delta_{r3}\end{bmatrix}\\
&= \det\begin{bmatrix}\delta_{i1} & \delta_{i2} & \delta_{i3}\\ \delta_{j1} & \delta_{j2} & \delta_{j3}\\ \delta_{k1} & \delta_{k2} & \delta_{k3}\end{bmatrix} \det\begin{bmatrix}\delta_{p1} & \delta_{q1} & \delta_{r1}\\ \delta_{p2} & \delta_{q2} & \delta_{r2}\\ \delta_{p3} & \delta_{q3} & \delta_{r3}\end{bmatrix} \qquad (\text{since } \det \boldsymbol{T} = \det(\boldsymbol{T}^T))\\
&= \det\left\{\begin{bmatrix}\delta_{i1} & \delta_{i2} & \delta_{i3}\\ \delta_{j1} & \delta_{j2} & \delta_{j3}\\ \delta_{k1} & \delta_{k2} & \delta_{k3}\end{bmatrix} \begin{bmatrix}\delta_{p1} & \delta_{q1} & \delta_{r1}\\ \delta_{p2} & \delta_{q2} & \delta_{r2}\\ \delta_{p3} & \delta_{q3} & \delta_{r3}\end{bmatrix}\right\} \qquad (\text{since } (\det \boldsymbol{R})(\det \boldsymbol{S}) = \det(\boldsymbol{RS}))\\
&= \det\begin{bmatrix}\delta_{im}\delta_{mp} & \delta_{im}\delta_{mq} & \delta_{im}\delta_{mr}\\ \delta_{jm}\delta_{mp} & \delta_{jm}\delta_{mq} & \delta_{jm}\delta_{mr}\\ \delta_{km}\delta_{mp} & \delta_{km}\delta_{mq} & \delta_{km}\delta_{mr}\end{bmatrix}\\
&= \det\begin{bmatrix}\delta_{ip} & \delta_{iq} & \delta_{ir}\\ \delta_{jp} & \delta_{jq} & \delta_{jr}\\ \delta_{kp} & \delta_{kq} & \delta_{kr}\end{bmatrix}. \qquad (1.15)
\end{aligned}
$$

From Eqn. (1.15) and the relation $\delta_{ii} = 3$, we obtain the following identities (the first of which is known as the ϵ–δ identity):

$$\epsilon_{ijk}\epsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}, \qquad (1.16a)$$

$$\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}, \qquad (1.16b)$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6. \qquad (1.16c)$$

Using Eqn. (1.9) and the ϵ–δ identity, we get

$$
\begin{aligned}
(\boldsymbol{u} \times \boldsymbol{v}) \cdot (\boldsymbol{u} \times \boldsymbol{v}) &= \epsilon_{ijk}\epsilon_{imn}u_j v_k u_m v_n\\
&= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})u_j v_k u_m v_n\\
&= (u_m v_n u_m v_n - u_j v_m u_m v_j)\\
&= (\boldsymbol{u} \cdot \boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{v}) - (\boldsymbol{u} \cdot \boldsymbol{v})^2. \qquad (1.17)
\end{aligned}
$$

The vector triple products $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})$ and $(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$, defined as the cross product of $\boldsymbol{u}$ with $\boldsymbol{v} \times \boldsymbol{w}$, and the cross product of $\boldsymbol{u} \times \boldsymbol{v}$ with $\boldsymbol{w}$, respectively, are different in general, and are given by

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}, \qquad (1.18a)$$

$$(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}. \tag{1.18b}$$

The first relation is proved by noting that

$$\begin{aligned}
\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) &= \epsilon_{ijk} u_j (\boldsymbol{v} \times \boldsymbol{w})_k \boldsymbol{e}_i \\
&= \epsilon_{ijk} \epsilon_{kmn} u_j v_m w_n \boldsymbol{e}_i \\
&= \epsilon_{kij} \epsilon_{kmn} u_j v_m w_n \boldsymbol{e}_i \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) u_j v_m w_n \boldsymbol{e}_i \\
&= (u_n w_n v_i - u_m v_m w_i) \boldsymbol{e}_i \\
&= (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}.
\end{aligned}$$

The second relation is proved in an analogous manner.

1.2 Second-Order Tensors

A second-order tensor is a linear transformation that maps vectors to vectors. We shall denote the set of second-order tensors by Lin. If $\boldsymbol{T}$ is a second-order tensor that maps a vector $\boldsymbol{u}$ to a vector $\boldsymbol{v}$, then we write it as

$$\boldsymbol{v} = \boldsymbol{T}\boldsymbol{u}. \tag{1.19}$$

$\boldsymbol{T}$ satisfies the property

$$\boldsymbol{T}(a\boldsymbol{x} + b\boldsymbol{y}) = a\boldsymbol{T}\boldsymbol{x} + b\boldsymbol{T}\boldsymbol{y}, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in V \text{ and } a, b \in \Re.$$

By choosing $a = 1$, $b = -1$, and $\boldsymbol{x} = \boldsymbol{y}$, we get

$$\boldsymbol{T}(\boldsymbol{0}) = \boldsymbol{0}.$$

From the definition of a second-order tensor, it follows that the sum of two second-order tensors defined by

$$(\boldsymbol{R} + \boldsymbol{S})\boldsymbol{u} := \boldsymbol{R}\boldsymbol{u} + \boldsymbol{S}\boldsymbol{u} \quad \forall \boldsymbol{u} \in V,$$

and the scalar multiple of $\boldsymbol{T}$ by $\alpha \in \Re$, defined by

$$(\alpha \boldsymbol{T})\boldsymbol{u} := \alpha(\boldsymbol{T}\boldsymbol{u}) \quad \forall \boldsymbol{u} \in V,$$

are both second-order tensors. The two second-order tensors $\boldsymbol{R}$ and $\boldsymbol{S}$ are said to be equal if

$$\boldsymbol{R}\boldsymbol{u} = \boldsymbol{S}\boldsymbol{u} \quad \forall \boldsymbol{u} \in V. \tag{1.20}$$

The above condition is equivalent to the condition

$$(\boldsymbol{v}, \boldsymbol{R}\boldsymbol{u}) = (\boldsymbol{v}, \boldsymbol{S}\boldsymbol{u}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{1.21}$$

To see this, note that if Eqn. (1.20) holds, then clearly Eqn. (1.21) holds. On the other hand, if Eqn. (1.21) holds, then using the bilinearity property of the inner product, we have

$$(\boldsymbol{v}, (\boldsymbol{R}\boldsymbol{u} - \boldsymbol{S}\boldsymbol{u})) = 0 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V.$$

Choosing $\boldsymbol{v} = \boldsymbol{R}\boldsymbol{u} - \boldsymbol{S}\boldsymbol{u}$, we get $|\boldsymbol{R}\boldsymbol{u} - \boldsymbol{S}\boldsymbol{u}| = 0$, which proves Eqn. (1.20).

If we define the function $\boldsymbol{I} : V \to V$ by

$$\boldsymbol{I}\boldsymbol{u} := \boldsymbol{u} \quad \forall \boldsymbol{u} \in V, \tag{1.22}$$

then it is clear that $\boldsymbol{I} \in \text{Lin}$. $\boldsymbol{I}$ is called as the *identity tensor*.

Choosing $\boldsymbol{u} = \boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ in Eqn. (1.19), we get three vectors that can be expressed as a linear combination of the base vectors $\boldsymbol{e}_i$ as

$$\begin{aligned} \boldsymbol{T}\boldsymbol{e}_1 &= \alpha_1 \boldsymbol{e}_1 + \alpha_2 \boldsymbol{e}_2 + \alpha_3 \boldsymbol{e}_3 \\ \boldsymbol{T}\boldsymbol{e}_2 &= \alpha_4 \boldsymbol{e}_1 + \alpha_5 \boldsymbol{e}_2 + \alpha_6 \boldsymbol{e}_3 \\ \boldsymbol{T}\boldsymbol{e}_3 &= \alpha_7 \boldsymbol{e}_1 + \alpha_8 \boldsymbol{e}_2 + \alpha_9 \boldsymbol{e}_3, \end{aligned} \tag{1.23}$$

where α_i, $i = 1$ to 9, are scalar constants. Renaming the α_i as T_{ij}, $i = 1, 3$, $j = 1, 3$, we get

$$\boldsymbol{T}\boldsymbol{e}_j = T_{ij} \boldsymbol{e}_i. \tag{1.24}$$

The elements T_{ij} are called the components of the tensor $\boldsymbol{T}$ with respect to the base vectors $\boldsymbol{e}_j$; as seen from Eqn. (1.24), T_{ij} is the component of $\boldsymbol{T}\boldsymbol{e}_j$ in the $\boldsymbol{e}_i$ direction. Taking the dot product of both sides of Eqn. (1.24) with $\boldsymbol{e}_k$ for some particular k, we get

$$\boldsymbol{e}_k \cdot \boldsymbol{T}\boldsymbol{e}_j = T_{ij} \delta_{ik} = T_{kj},$$

or, replacing k by i,

$$T_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{T}\boldsymbol{e}_j. \tag{1.25}$$

By choosing $\boldsymbol{v} = \boldsymbol{e}_i$ and $\boldsymbol{u} = \boldsymbol{e}_j$ in Eqn. (1.21), it is clear that the components of two equal tensors are equal. From Eqn. (1.25), the components of the identity tensor in any orthonormal coordinate system $\boldsymbol{e}_i$ are

$$I_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{I}\boldsymbol{e}_j = \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}. \tag{1.26}$$

Thus, the components of the identity tensor are scalars that are independent of the Cartesian basis. Using Eqn. (1.24), we write Eqn. (1.19) in component form (where the components are with respect to a particular orthonormal basis $\{\boldsymbol{e}_i\}$) as

$$v_i \boldsymbol{e}_i = \boldsymbol{T}(u_j \boldsymbol{e}_j) = u_j \boldsymbol{T}\boldsymbol{e}_j = u_j T_{ij} \boldsymbol{e}_i,$$

which, by virtue of the uniqueness of the components of any element of a vector space, yields

$$v_i = T_{ij} u_j. \tag{1.27}$$

Thus, the components of the vector $\boldsymbol{v}$ are obtained by a matrix multiplication of the components of $\boldsymbol{T}$, and the components of $\boldsymbol{u}$.

The transpose of $\boldsymbol{T}$, denoted by $\boldsymbol{T}^T$, is defined using the inner product as

$$(\boldsymbol{T}^T \boldsymbol{u}, \boldsymbol{v}) := (\boldsymbol{u}, \boldsymbol{T}\boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{1.28}$$

Once again, it follows from the definition that $\boldsymbol{T}^T$ is a second-order tensor. The transpose has the following properties:

$$\begin{aligned} (\boldsymbol{T}^T)^T &= \boldsymbol{T}, \\ (\alpha \boldsymbol{T})^T &= \alpha \boldsymbol{T}^T, \\ (\boldsymbol{R} + \boldsymbol{S})^T &= \boldsymbol{R}^T + \boldsymbol{S}^T. \end{aligned}$$

If (T_{ij}) represent the components of the tensor $\boldsymbol{T}$, then the components of $\boldsymbol{T}^T$ are

$$\begin{aligned} (\boldsymbol{T}^T)_{ij} &= \boldsymbol{e}_i \cdot \boldsymbol{T}^T \boldsymbol{e}_j \\ &= \boldsymbol{T}\boldsymbol{e}_i \cdot \boldsymbol{e}_j \\ &= T_{ji}. \end{aligned} \tag{1.29}$$

The tensor $\boldsymbol{T}$ is said to be symmetric if

$$\boldsymbol{T}^T = \boldsymbol{T},$$

and skew-symmetric (or anti-symmetric) if

$$\boldsymbol{T}^T = -\boldsymbol{T}.$$

Any tensor $\boldsymbol{T}$ can be decomposed uniquely into a symmetric and an skew-symmetric part as

$$\boldsymbol{T} = \boldsymbol{T}_s + \boldsymbol{T}_{ss}, \tag{1.30}$$

where

$$\begin{aligned} \boldsymbol{T}_s &= \frac{1}{2}(\boldsymbol{T} + \boldsymbol{T}^T), \\ \boldsymbol{T}_{ss} &= \frac{1}{2}(\boldsymbol{T} - \boldsymbol{T}^T). \end{aligned}$$

The product of two second-order tensors $\boldsymbol{R}\boldsymbol{S}$ is the composition of the two operations $\boldsymbol{R}$ and $\boldsymbol{S}$, with $\boldsymbol{S}$ operating first, and defined by the relation

$$(\boldsymbol{R}\boldsymbol{S})\boldsymbol{u} := \boldsymbol{R}(\boldsymbol{S}\boldsymbol{u}) \quad \forall \boldsymbol{u} \in V. \tag{1.31}$$

Since $\boldsymbol{RS}$ is a linear transformation that maps vectors to vectors, we conclude that the product of two second-order tensors is also a second-order tensor. From the definition of the identity tensor given by (1.22) it follows that $\boldsymbol{RI} = \boldsymbol{IR} = \boldsymbol{R}$. If $\boldsymbol{T}$ represents the product $\boldsymbol{RS}$, then its components are given by

$$\begin{aligned} T_{ij} &= \boldsymbol{e}_i \cdot (\boldsymbol{RS})\boldsymbol{e}_j \\ &= \boldsymbol{e}_i \cdot \boldsymbol{R}(\boldsymbol{S}\boldsymbol{e}_j) \\ &= \boldsymbol{e}_i \cdot \boldsymbol{R}(S_{kj}\boldsymbol{e}_k) \\ &= \boldsymbol{e}_i \cdot S_{kj}\boldsymbol{R}\boldsymbol{e}_k \\ &= S_{kj}(\boldsymbol{e}_i \cdot \boldsymbol{R}\boldsymbol{e}_k) \\ &= S_{kj}R_{ik} \\ &= R_{ik}S_{kj}, \end{aligned} \tag{1.32}$$

which is consistent with matrix multiplication. Also consistent with the results from matrix theory, we have $(\boldsymbol{RS})^T = \boldsymbol{S}^T\boldsymbol{R}^T$, which follows from Eqns. (1.21), (1.28) and (1.31).

1.2.1 The tensor product

We now introduce the concept of a tensor product, which is convenient for working with tensors of rank higher than two. We first define the dyadic or tensor product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ by

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} := (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \quad \forall \boldsymbol{c} \in V. \tag{1.33}$$

Note that the tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$ cannot be defined except in terms of its operation on a vector $\boldsymbol{c}$. We now prove that $\boldsymbol{a} \otimes \boldsymbol{b}$ defines a second-order tensor. The above rule obviously maps a vector into another vector. All that we need to do is to prove that it is a linear map. For arbitrary scalars c and d, and arbitrary vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we have

$$\begin{aligned} (\boldsymbol{a} \otimes \boldsymbol{b})(c\boldsymbol{x} + d\boldsymbol{y}) &= [\boldsymbol{b} \cdot (c\boldsymbol{x} + d\boldsymbol{y})]\,\boldsymbol{a} \\ &= [c\boldsymbol{b} \cdot \boldsymbol{x} + d\boldsymbol{b} \cdot \boldsymbol{y}]\,\boldsymbol{a} \\ &= c(\boldsymbol{b} \cdot \boldsymbol{x})\boldsymbol{a} + d(\boldsymbol{b} \cdot \boldsymbol{y})\boldsymbol{a} \\ &= c[(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{x}] + d\,[(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{y}]\,, \end{aligned}$$

which proves that $\boldsymbol{a} \otimes \boldsymbol{b}$ is a linear function. Hence, $\boldsymbol{a} \otimes \boldsymbol{b}$ is a second-order tensor. Any second-order tensor $\boldsymbol{T}$ can be written as

$$\boldsymbol{T} = T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j, \tag{1.34}$$

where the components of the tensor, T_{ij} are given by Eqn. (1.25). To see this, we consider the action of $\boldsymbol{T}$ on an arbitrary vector $\boldsymbol{u}$:

$$\begin{aligned}\boldsymbol{T}\boldsymbol{u} = (\boldsymbol{T}\boldsymbol{u})_i\boldsymbol{e}_i &= [\boldsymbol{e}_i \cdot (\boldsymbol{T}\boldsymbol{u})]\,\boldsymbol{e}_i \\ &= \{\boldsymbol{e}_i \cdot [\boldsymbol{T}(u_j\boldsymbol{e}_j)]\}\,\boldsymbol{e}_i \\ &= \{u_j\,[\boldsymbol{e}_i \cdot (\boldsymbol{T}\boldsymbol{e}_j)]\}\,\boldsymbol{e}_i \\ &= \{(\boldsymbol{u}\cdot\boldsymbol{e}_j)\,[\boldsymbol{e}_i \cdot (\boldsymbol{T}\boldsymbol{e}_j)]\}\,\boldsymbol{e}_i \\ &= [\boldsymbol{e}_i \cdot (\boldsymbol{T}\boldsymbol{e}_j)]\,[(\boldsymbol{u}\cdot\boldsymbol{e}_j)\boldsymbol{e}_i] \\ &= [\boldsymbol{e}_i \cdot (\boldsymbol{T}\boldsymbol{e}_j)]\,[(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\boldsymbol{u}] \\ &= \{[\boldsymbol{e}_i \cdot (\boldsymbol{T}\boldsymbol{e}_j)]\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\}\,\boldsymbol{u}.\end{aligned}$$

Hence, we conclude that any second-order tensor admits the representation given by Eqn. (1.34), with the nine components T_{ij}, $i = 1, 2, 3$, $j = 1, 2, 3$, given by Eqn. (1.25).

From Eqns. (1.26) and (1.34), it follows that

$$\boldsymbol{I} = \bar{\boldsymbol{e}}_i \otimes \bar{\boldsymbol{e}}_i, \tag{1.35}$$

where $\{\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \bar{\boldsymbol{e}}_3\}$ is *any* orthonormal coordinate frame. If $\boldsymbol{T}$ is represented as given by Eqn. (1.34), it follows from Eqn. (1.29) that the transpose of $\boldsymbol{T}$ can be represented as

$$\boldsymbol{T}^T = T_{ji}\boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{1.36}$$

From Eqns. (1.34) and (1.36), we deduce that a tensor is symmetric ($\boldsymbol{T} = \boldsymbol{T}^T$) if and only if $T_{ij} = T_{ji}$ for all possible i and j. We now show how all the properties of a second-order tensor derived so far can be derived using the dyadic product.

Using Eqn. (1.24), we see that the components of a dyad $\boldsymbol{a}\otimes\boldsymbol{b}$ are given by

$$\begin{aligned}(\boldsymbol{a}\otimes\boldsymbol{b})_{ij} &= \boldsymbol{e}_i \cdot (\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{e}_j \\ &= \boldsymbol{e}_i \cdot (\boldsymbol{b}\cdot\boldsymbol{e}_j)\boldsymbol{a} \\ &= a_i b_j.\end{aligned} \tag{1.37}$$

Using the above component form, one can easily verify that

$$(\boldsymbol{a}\otimes\boldsymbol{b})(\boldsymbol{c}\otimes\boldsymbol{d}) = (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}\otimes\boldsymbol{d}, \tag{1.38}$$

$$\boldsymbol{T}(\boldsymbol{a}\otimes\boldsymbol{b}) = (\boldsymbol{T}\boldsymbol{a})\otimes\boldsymbol{b}, \tag{1.39}$$

$$(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{T} = \boldsymbol{a}\otimes(\boldsymbol{T}^T\boldsymbol{b}). \tag{1.40}$$

The components of a vector $\boldsymbol{v}$ obtained by a second-order tensor $\boldsymbol{T}$ operating on a vector $\boldsymbol{u}$ are obtained by noting that

$$v_i\boldsymbol{e}_i = T_{ij}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\boldsymbol{u} = T_{ij}(\boldsymbol{u}\cdot\boldsymbol{e}_j)\boldsymbol{e}_i = T_{ij}u_j\boldsymbol{e}_i, \tag{1.41}$$

which in equivalent to Eqn. (1.27).

1.2.2 Cofactor of a tensor

In order to define the concept of a inverse of a tensor in a later section, it is convenient to first introduce the cofactor tensor, denoted by $\mathbf{cof}\,\boldsymbol{T}$, and defined by the relation

$$(\mathbf{cof}\,\boldsymbol{T})_{ij} = \frac{1}{2}\epsilon_{imn}\epsilon_{jpq}T_{mp}T_{nq}. \tag{1.42}$$

Equation (1.42) when written out explicitly reads

$$[\mathbf{cof}\,\boldsymbol{T}] = \begin{bmatrix} T_{22}T_{33} - T_{23}T_{32} & T_{23}T_{31} - T_{21}T_{33} & T_{21}T_{32} - T_{22}T_{31} \\ T_{32}T_{13} - T_{33}T_{12} & T_{33}T_{11} - T_{31}T_{13} & T_{31}T_{12} - T_{32}T_{11} \\ T_{12}T_{23} - T_{13}T_{22} & T_{13}T_{21} - T_{11}T_{23} & T_{11}T_{22} - T_{12}T_{21} \end{bmatrix}. \tag{1.43}$$

It follows from the definition in Eqn. (1.42) that

$$\mathbf{cof}\,\boldsymbol{T}(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{T}\boldsymbol{u}\times\boldsymbol{T}\boldsymbol{v} \quad \forall \boldsymbol{u},\boldsymbol{v}\in V. \tag{1.44}$$

By using Eqns. (1.15) and (1.42), we also get the following explicit formula for the cofactor:

$$(\mathbf{cof}\,\boldsymbol{T})^T = \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{T})^2 - \operatorname{tr}(\boldsymbol{T}^2)\right]\boldsymbol{I} - (\operatorname{tr}\boldsymbol{T})\boldsymbol{T} + \boldsymbol{T}^2. \tag{1.45}$$

It immediately follows from Eqn. (1.45) that $\mathbf{cof}\,\boldsymbol{T}$ corresponding to a given $\boldsymbol{T}$ is unique. We also observe that

$$\mathbf{cof}\,\boldsymbol{T}^T = (\mathbf{cof}\,\boldsymbol{T})^T, \tag{1.46}$$

and that

$$(\mathbf{cof}\,\boldsymbol{T})^T\boldsymbol{T} = \boldsymbol{T}(\mathbf{cof}\,\boldsymbol{T})^T. \tag{1.47}$$

Similar to the result for determinants, the cofactor of the product of two tensors is the product of the cofactors of the tensors, i.e.,

$$\mathbf{cof}\,(\boldsymbol{R}\boldsymbol{S}) = (\mathbf{cof}\,\boldsymbol{R})(\mathbf{cof}\,\boldsymbol{S}).$$

The above result can be proved using Eqn. (1.42).

1.2.3 Principal invariants of a second-order tensor

The principal invariants of a tensor $\boldsymbol{T}$ are defined as

$$I_1 = \operatorname{tr}\boldsymbol{T} = T_{ii}, \tag{1.48a}$$

$$I_2 = \operatorname{tr}\mathbf{cof}\,\boldsymbol{T} = \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{T})^2 - \operatorname{tr}\boldsymbol{T}^2\right], \tag{1.48b}$$

$$I_3 = \det \boldsymbol{T} = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}T_{pi}T_{qj}T_{rk} = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}T_{ip}T_{jq}T_{kr}, \tag{1.48c}$$

The first and third invariants are referred to as the trace and determinant of $\boldsymbol{T}$.

The scalars I_1, I_2 and I_3 are called the *principal* invariants of $\boldsymbol{T}$. The reason for calling them invariant is that they do not depend on the basis; i.e., although the individual components of $\boldsymbol{T}$ change with a change in basis, I_1, I_2 and I_3 remain the same as we show in Section 1.4. The reason for calling them as the principal invariants is that any other scalar invariant of $\boldsymbol{T}$ can be expressed in terms of them.

From Eqn. (1.48a), it is clear that the trace is a linear operation, i.e.,

$$\operatorname{tr}(\alpha \boldsymbol{R} + \beta \boldsymbol{S}) = \alpha \operatorname{tr} \boldsymbol{R} + \beta \operatorname{tr} \boldsymbol{S} \quad \forall \alpha, \beta \in \Re \text{ and } \boldsymbol{R}, \boldsymbol{S} \in \text{Lin}.$$

It also follows that

$$\operatorname{tr} \boldsymbol{T}^T = \operatorname{tr} \boldsymbol{T}. \tag{1.49}$$

By letting $\boldsymbol{T} = \boldsymbol{a} \otimes \boldsymbol{b}$ in Eqn. (1.48a), we obtain

$$\operatorname{tr}(\boldsymbol{a} \otimes \boldsymbol{b}) = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i = \boldsymbol{a} \cdot \boldsymbol{b}. \tag{1.50}$$

Using the linearity of the trace operator, and Eqn. (1.34), we get

$$\operatorname{tr} \boldsymbol{T} = \operatorname{tr}\,(T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = T_{ij} \operatorname{tr}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = T_{ij}\boldsymbol{e}_i \cdot \boldsymbol{e}_j = T_{ii},$$

which agrees with Eqn. (1.48a). One can easily prove using indicial notation that

$$\operatorname{tr}(\boldsymbol{RS}) = \operatorname{tr}(\boldsymbol{SR}).$$

Similar to the vector inner product given by Eqn. (1.4), we can define a tensor inner product of two second-order tensors $\boldsymbol{R}$ and $\boldsymbol{S}$, denoted by $\boldsymbol{R} : \boldsymbol{S}$, by

$$(\boldsymbol{R}, \boldsymbol{S}) = \boldsymbol{R} : \boldsymbol{S} := \operatorname{tr}(\boldsymbol{R}^T\boldsymbol{S}) = \operatorname{tr}(\boldsymbol{R}\boldsymbol{S}^T) = \operatorname{tr}(\boldsymbol{S}\boldsymbol{R}^T) = \operatorname{tr}(\boldsymbol{S}^T\boldsymbol{R}) = R_{ij}S_{ij}. \tag{1.51}$$

We have the following useful property:

$$\boldsymbol{R} : (\boldsymbol{ST}) = (\boldsymbol{S}^T\boldsymbol{R}) : \boldsymbol{T} = (\boldsymbol{R}\boldsymbol{T}^T) : \boldsymbol{S} = (\boldsymbol{T}\boldsymbol{R}^T) : \boldsymbol{S}^T, \tag{1.52}$$

since

$$\begin{aligned} \boldsymbol{R} : (\boldsymbol{ST}) &= \operatorname{tr}(\boldsymbol{ST})^T\boldsymbol{R} = \operatorname{tr} \boldsymbol{T}^T(\boldsymbol{S}^T\boldsymbol{R}) = (\boldsymbol{S}^T\boldsymbol{R}) : \boldsymbol{T} = (\boldsymbol{R}^T\boldsymbol{S}) : \boldsymbol{T}^T \\ &= \operatorname{tr} \boldsymbol{S}^T(\boldsymbol{R}\boldsymbol{T}^T) = (\boldsymbol{R}\boldsymbol{T}^T) : \boldsymbol{S} = (\boldsymbol{T}\boldsymbol{R}^T) : \boldsymbol{S}^T. \end{aligned}$$

The second equality in Eqn. (1.48b) follows by taking the trace of either side of Eqn. (1.45).

From Eqn. (1.48c), it can be seen that

$$\det \boldsymbol{I} = 1, \tag{1.53a}$$
$$\det \boldsymbol{T} = \det \boldsymbol{T}^T, \tag{1.53b}$$
$$\det(\alpha \boldsymbol{T}) = \alpha^3 \det \boldsymbol{T}, \tag{1.53c}$$
$$\det(\boldsymbol{RS}) = (\det \boldsymbol{R})(\det \boldsymbol{S}) = \det(\boldsymbol{SR}), \tag{1.53d}$$
$$\det(\boldsymbol{R} + \boldsymbol{S}) = \det \boldsymbol{R} + \mathbf{cof}\, \boldsymbol{R} : \boldsymbol{S} + \boldsymbol{R} : \mathbf{cof}\, \boldsymbol{S} + \det \boldsymbol{S}. \tag{1.53e}$$

By using Eqns. (1.15), (1.48c) and (1.53b), we also have

$$\epsilon_{pqr}(\det \boldsymbol{T}) = \epsilon_{ijk} T_{ip} T_{jq} T_{kr} = \epsilon_{ijk} T_{pi} T_{qj} T_{rk}. \tag{1.54}$$

By choosing $(p, q, r) = (1, 2, 3)$ in the above equation, we get

$$\det \boldsymbol{T} = \epsilon_{ijk} T_{i1} T_{j2} T_{k3} = \epsilon_{ijk} T_{1i} T_{2j} T_{3k}.$$

Using Eqns. (1.42) and (1.54), we get

$$\boldsymbol{T}(\mathbf{cof}\, \boldsymbol{T})^T = (\mathbf{cof}\, \boldsymbol{T})^T \boldsymbol{T} = (\det \boldsymbol{T})\boldsymbol{I}. \tag{1.55}$$

We now state the following important theorem without proof:

Theorem 1.2.1. *Given a tensor $\boldsymbol{T}$, there exists a nonzero vector $\boldsymbol{n}$ such that $\boldsymbol{Tn} = \mathbf{0}$ if and only if* $\det \boldsymbol{T} = 0$.

1.2.4 Inverse of a tensor

The inverse of a second-order tensor $\boldsymbol{T}$, denoted by $\boldsymbol{T}^{-1}$, is defined by

$$\boldsymbol{T}^{-1}\boldsymbol{T} = \boldsymbol{I}, \tag{1.56}$$

where $\boldsymbol{I}$ is the identity tensor. A characterization of an invertible tensor is the following:

Theorem 1.2.2. *A tensor $\boldsymbol{T}$ is invertible if and only if* $\det \boldsymbol{T} \neq 0$. *The inverse, if it exists, is unique.*

Proof. Assuming $\boldsymbol{T}^{-1}$ exists, from Eqns. (1.53d) and (1.56), we have $(\det \boldsymbol{T})(\det \boldsymbol{T}^{-1}) = 1$, and hence $\det \boldsymbol{T} \neq 0$.

Conversely, if $\det \boldsymbol{T} \neq 0$, then from Eqn. (1.55), we see that at least one inverse exists, and is given by

$$\boldsymbol{T}^{-1} = \frac{1}{\det \boldsymbol{T}}(\mathbf{cof}\, \boldsymbol{T})^T. \tag{1.57}$$

Let $\boldsymbol{T}_1^{-1}$ and $\boldsymbol{T}_2^{-1}$ be two inverses that satisfy $\boldsymbol{T}_1^{-1}\boldsymbol{T} = \boldsymbol{T}_2^{-1}\boldsymbol{T} = \boldsymbol{I}$, from which it follows that $(\boldsymbol{T}_1^{-1} - \boldsymbol{T}_2^{-1})\boldsymbol{T} = \mathbf{0}$. Choose $\boldsymbol{T}_2^{-1}$ to be given by the expression in Eqn. (1.57) so that, by virtue of Eqn. (1.55), we also have $\boldsymbol{T}\boldsymbol{T}_2^{-1} = \boldsymbol{I}$. Multiplying both sides of $(\boldsymbol{T}_1^{-1} - \boldsymbol{T}_2^{-1})\boldsymbol{T} = \mathbf{0}$ by $\boldsymbol{T}_2^{-1}$, we get $\boldsymbol{T}_1^{-1} = \boldsymbol{T}_2^{-1}$, which establishes the uniqueness of $\boldsymbol{T}^{-1}$. □

Thus, if $\boldsymbol{T}$ is invertible, then from Eqn. (1.55), we get

$$\mathbf{cof}\,\boldsymbol{T} = (\det \boldsymbol{T})\boldsymbol{T}^{-T}. \tag{1.58}$$

From Eqns. (1.55) and (1.57), we have

$$\boldsymbol{T}^{-1}\boldsymbol{T} = \boldsymbol{T}\boldsymbol{T}^{-1} = \boldsymbol{I}. \tag{1.59}$$

If $\boldsymbol{T}$ is invertible, we have

$$\boldsymbol{T}\boldsymbol{u} = \boldsymbol{v} \iff \boldsymbol{u} = \boldsymbol{T}^{-1}\boldsymbol{v}, \quad \boldsymbol{u}, \boldsymbol{v} \in V.$$

By the above property, $\boldsymbol{T}^{-1}$ clearly maps vectors to vectors. Hence, to prove that $\boldsymbol{T}^{-1}$ is a second-order tensor, we just need to prove linearity. Let $\boldsymbol{a}, \boldsymbol{b} \in V$ be two arbitrary vectors, and let $\boldsymbol{u} = \boldsymbol{T}^{-1}\boldsymbol{a}$ and $\boldsymbol{v} = \boldsymbol{T}^{-1}\boldsymbol{b}$. Since $\boldsymbol{I} = \boldsymbol{T}^{-1}\boldsymbol{T}$, we have

$$\begin{aligned} \boldsymbol{I}(\alpha\boldsymbol{u} + \beta\boldsymbol{v}) &= \boldsymbol{T}^{-1}\boldsymbol{T}(\alpha\boldsymbol{u} + \beta\boldsymbol{v}) \\ &= \boldsymbol{T}^{-1}[\boldsymbol{T}(\alpha\boldsymbol{u} + \beta\boldsymbol{v})] \\ &= \boldsymbol{T}^{-1}[\alpha\boldsymbol{T}\boldsymbol{u} + \beta\boldsymbol{T}\boldsymbol{v}] \\ &= \boldsymbol{T}^{-1}(\alpha\boldsymbol{a} + \beta\boldsymbol{b}), \end{aligned}$$

which implies that

$$\boldsymbol{T}^{-1}(\alpha\boldsymbol{a} + \beta\boldsymbol{b}) = \alpha\boldsymbol{T}^{-1}\boldsymbol{a} + \beta\boldsymbol{T}^{-1}\boldsymbol{b} \quad \forall \boldsymbol{a}, \boldsymbol{b} \in V \text{ and } \alpha, \beta \in \Re.$$

The inverse of the product of two invertible tensors $\boldsymbol{R}$ and $\boldsymbol{S}$ is

$$(\boldsymbol{R}\boldsymbol{S})^{-1} = \boldsymbol{S}^{-1}\boldsymbol{R}^{-1}, \tag{1.60}$$

since the inverse is unique, and

$$\boldsymbol{S}^{-1}\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{S} = \boldsymbol{S}^{-1}\boldsymbol{I}\boldsymbol{S} = \boldsymbol{S}^{-1}\boldsymbol{S} = \boldsymbol{I}.$$

Similarly, if $\boldsymbol{T}$ is invertible, then $\boldsymbol{T}^T$ is invertible since $\det \boldsymbol{T}^T = \det \boldsymbol{T} \neq 0$. The inverse of the transpose is given by

$$(\boldsymbol{T}^T)^{-1} = (\boldsymbol{T}^{-1})^T, \tag{1.61}$$

since

$$(\boldsymbol{T}^{-1})^T\boldsymbol{T}^T = (\boldsymbol{T}\boldsymbol{T}^{-1})^T = \boldsymbol{I}^T = \boldsymbol{I}.$$

Hence, without fear of ambiguity, we can write

$$\boldsymbol{T}^{-T} := (\boldsymbol{T}^T)^{-1} = (\boldsymbol{T}^{-1})^T.$$

From Eqn. (1.61), it follows that if $\boldsymbol{T} \in \mathrm{Sym}$, then $\boldsymbol{T}^{-1} \in \mathrm{Sym}$.

1.2.5 Eigenvalues and eigenvectors of tensors

If $\boldsymbol{T}$ is an arbitrary tensor, a vector $\boldsymbol{n}$ is said to be an eigenvector of $\boldsymbol{T}$ if there exists λ such that

$$\boldsymbol{T}\boldsymbol{n} = \lambda\boldsymbol{n}. \tag{1.62}$$

Writing the above equation as $(\boldsymbol{T}-\lambda\boldsymbol{I})\boldsymbol{n} = \boldsymbol{0}$, we see from Theorem 1.2.1 that a nontrivial eigenvector $\boldsymbol{n}$ exists if and only if

$$\det(\boldsymbol{T}-\lambda\boldsymbol{I}) = 0.$$

This is known as the *characteristic* equation of $\boldsymbol{T}$. Using Eqn. (1.53e), the characteristic equation can be written as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0, \tag{1.63}$$

where I_1, I_2 and I_3 are the principal invariants given by Eqns. (1.48). Since the principal invariants are real, Eqn. (1.63) has either one or three real roots. If one of the eigenvalues is complex, then it follows from Eqn. (1.62) that the corresponding eigenvector is also complex. By taking the complex conjugate of both sides of Eqn. (1.62), we see that the complex conjugate of the complex eigenvalue, and the corresponding complex eigenvector are also eigenvalues and eigenvectors, respectively. Thus, eigenvalues and eigenvectors, if complex, occur in complex conjugate pairs. If λ_1, λ_2, λ_3 are the roots of the characteristic equation, then from Eqns. (1.48) and (1.63), it follows that

$$\begin{aligned}
I_1 &= \operatorname{tr}\boldsymbol{T} = T_{11}+T_{22}+T_{33}, \\
&= \lambda_1+\lambda_2+\lambda_3, && \text{(1.64a)}\\
I_2 &= \operatorname{tr}\mathbf{cof}\,\boldsymbol{T} = \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{T})^2 - \operatorname{tr}(\boldsymbol{T}^2)\right] = \begin{vmatrix} T_{11} & T_{12}\\ T_{21} & T_{22}\end{vmatrix} + \begin{vmatrix} T_{22} & T_{23}\\ T_{32} & T_{33}\end{vmatrix} + \begin{vmatrix} T_{11} & T_{13}\\ T_{31} & T_{33}\end{vmatrix}\\
&= \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3, && \text{(1.64b)}\\
I_3 &= \det(\boldsymbol{T}) = \frac{1}{6}\left[(\operatorname{tr}\boldsymbol{T})^3 - 3(\operatorname{tr}\boldsymbol{T})(\operatorname{tr}\boldsymbol{T}^2) + 2\operatorname{tr}\boldsymbol{T}^3\right] = \epsilon_{ijk}T_{i1}T_{j2}T_{k3}\\
&= \lambda_1\lambda_2\lambda_3, && \text{(1.64c)}
\end{aligned}$$

where $|.|$ denotes the determinant. The set of eigenvalues $\{\lambda_1,\lambda_2,\lambda_3\}$ is known as the spectrum of $\boldsymbol{T}$. The expression for the determinant in Eqn. (1.64c) is derived in Eqn. (1.66) below.

If λ is an eigenvalue, and $\boldsymbol{n}$ is the associated eigenvector of $\boldsymbol{T}$, then λ^2 is the eigenvalue of $\boldsymbol{T}^2$, and $\boldsymbol{n}$ is the associated eigenvector, since

$$\boldsymbol{T}^2\boldsymbol{n} = \boldsymbol{T}(\boldsymbol{T}\boldsymbol{n}) = \boldsymbol{T}(\lambda\boldsymbol{n}) = \lambda\boldsymbol{T}\boldsymbol{n} = \lambda^2\boldsymbol{n}.$$

In general, λ^n is an eigenvalue of $\boldsymbol{T}^n$ with associated eigenvector $\boldsymbol{n}$. The eigenvalues of $\boldsymbol{T}^T$ and $\boldsymbol{T}$ are the same since their characteristic equations are the same.

An extremely important result is the following:

Theorem 1.2.3 (Cayley–Hamilton Theorem)**.** *A tensor $\boldsymbol{T}$ satisfies an equation having the same form as its characteristic equation, i.e.,*

$$\boldsymbol{T}^3 - I_1\boldsymbol{T}^2 + I_2\boldsymbol{T} - I_3\boldsymbol{I} = \boldsymbol{0} \quad \forall \boldsymbol{T}. \tag{1.65}$$

Proof. Multiplying Eqn. (1.45) by $\boldsymbol{T}$, we get

$$(\mathbf{cof}\,\boldsymbol{T})^T\boldsymbol{T} = I_2\boldsymbol{T} - I_1\boldsymbol{T}^2 + \boldsymbol{T}^3.$$

Since by Eqn. (1.55), $(\mathbf{cof}\,\boldsymbol{T})^T\boldsymbol{T} = (\det\boldsymbol{T})\boldsymbol{I} = I_3\boldsymbol{I}$, the result follows. □

By taking the trace of both sides of Eqn. (1.65), and using Eqn. (1.48b), we get

$$\det\boldsymbol{T} = \frac{1}{6}\left[(\operatorname{tr}\boldsymbol{T})^3 - 3(\operatorname{tr}\boldsymbol{T})(\operatorname{tr}\boldsymbol{T}^2) + 2\operatorname{tr}\boldsymbol{T}^3\right]. \tag{1.66}$$

From the above expression and the properties of the trace operator, Eqn. (1.53b) follows.

We have

$$\lambda_i = 0,\ i = 1,2,3 \iff \mathcal{I}_{\boldsymbol{T}} = \boldsymbol{0} \iff \operatorname{tr}(\boldsymbol{T}) = \operatorname{tr}(\boldsymbol{T}^2) = \operatorname{tr}(\boldsymbol{T}^3) = 0. \tag{1.67}$$

The proof is as follows. If all the invariants are zero, then from the characteristic equation given by Eqn. (1.63), it follows that all the eigenvalues are zero. If all the eigenvalues are zero, then from Eqns. (1.64), it follows that all the principal invariants $\mathcal{I}_{\boldsymbol{T}}$ are zero. If $\operatorname{tr}(\boldsymbol{T}) = \operatorname{tr}(\boldsymbol{T}^2) = \operatorname{tr}(\boldsymbol{T}^3) = 0$, then again from Eqns. (1.64) it follows that the principal invariants are zero. Conversely, if all the principal invariants are zero, then all the eigenvalues are zero from which it follows that $\operatorname{tr}\boldsymbol{T}^j = \sum_{i=1}^3 \lambda_i^j$, $j = 1,2,3$ are zero.

Consider the second-order tensor $\boldsymbol{u}\otimes\boldsymbol{v}$. By Eqn. (1.42), it follows that

$$\mathbf{cof}\,(\boldsymbol{u}\otimes\boldsymbol{v}) = \boldsymbol{0}, \tag{1.68}$$

so that the second invariant, which is the trace of the above tensor, is zero. Similarly, on using Eqn. (1.48c), we get the third invariant as zero. The first invariant is given by $\boldsymbol{u}\cdot\boldsymbol{v}$. Thus, from the characteristic equation, it follows that the eigenvalues of $\boldsymbol{u}\otimes\boldsymbol{v}$ are $(0, 0, \boldsymbol{u}\cdot\boldsymbol{v})$. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular, $\boldsymbol{u}\otimes\boldsymbol{v}$ is an example of a nonzero tensor all of whose eigenvalues are zero.

1.3 Skew-Symmetric Tensors

Let $\boldsymbol{W} \in \text{Skw}$ and let $\boldsymbol{u}, \boldsymbol{v} \in V$. Then

$$(\boldsymbol{u}, \boldsymbol{W}\boldsymbol{v}) = (\boldsymbol{W}^T\boldsymbol{u}, \boldsymbol{v}) = -(\boldsymbol{W}\boldsymbol{u}, \boldsymbol{v}) = -(\boldsymbol{v}, \boldsymbol{W}\boldsymbol{u}). \tag{1.69}$$

On setting $\boldsymbol{v} = \boldsymbol{u}$, we get

$$(\boldsymbol{u}, \boldsymbol{W}\boldsymbol{u}) = -(\boldsymbol{u}, \boldsymbol{W}\boldsymbol{u}),$$

which implies that

$$(\boldsymbol{u}, \boldsymbol{W}\boldsymbol{u}) = 0. \tag{1.70}$$

Thus, $\boldsymbol{W}\boldsymbol{u}$ is always orthogonal to $\boldsymbol{u}$ for any arbitrary vector $\boldsymbol{u}$. By choosing $\boldsymbol{u} = \boldsymbol{e}_i$ and $\boldsymbol{v} = \boldsymbol{e}_j$, we see from the above results that any skew-symmetric tensor $\boldsymbol{W}$ has only three independent components (in each coordinate frame), which suggests that it might be replaced by a vector. This observation leads us to the following result (which we state without proof):

Theorem 1.3.1. *Given any skew-symmetric tensor $\boldsymbol{W}$, there exists a unique vector $\boldsymbol{w}$, known as the* axial vector *or* dual vector*, corresponding to $\boldsymbol{W}$ such that*

$$\boldsymbol{W}\boldsymbol{u} = \boldsymbol{w} \times \boldsymbol{u} \quad \forall \boldsymbol{u} \in V. \tag{1.71}$$

Conversely, given any vector $\boldsymbol{w}$, there exists a unique skew-symmetric second-order tensor $\boldsymbol{W}$ such that Eqn. (1.71) *holds.*

Note that $\boldsymbol{W}\boldsymbol{u} = \boldsymbol{0}$ if and only if $\boldsymbol{u} = \alpha\boldsymbol{w}$, $\alpha \in \Re$. This result justifies the use of the terminology 'axial vector' used for $\boldsymbol{w}$. Also note that by virtue of the uniqueness of $\boldsymbol{w}$, the vector $\alpha\boldsymbol{w}$, $\alpha \in \Re$, is a one-dimensional subspace of V.

By choosing $\boldsymbol{u} = \boldsymbol{e}_j$ and taking the dot product of both sides with $\boldsymbol{e}_i$, Eqn. (1.71) can be expressed in component form as

$$\begin{aligned} W_{ij} &= -\epsilon_{ijk} w_k, \\ w_i &= -\frac{1}{2}\epsilon_{ijk} W_{jk}. \end{aligned} \tag{1.72}$$

More explicitly, if $\boldsymbol{w} = (w_1, w_2, w_3)$, then

$$\boldsymbol{W} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}.$$

From Eqn. (1.72), it follows that $\boldsymbol{W} : \boldsymbol{W} = -\text{tr}\,(\boldsymbol{W}^2) = 2\boldsymbol{w} \cdot \boldsymbol{w}$. Since $\text{tr}\,\boldsymbol{W} = \text{tr}\,\boldsymbol{W}^T = -\text{tr}\,\boldsymbol{W}$ and $\det \boldsymbol{W} = \det \boldsymbol{W}^T = -\det \boldsymbol{W}$, we have $\text{tr}\,\boldsymbol{W} = \det \boldsymbol{W} = 0$. The second invariant is given by $I_2 = [(\text{tr}\,\boldsymbol{W})^2 - \text{tr}\,(\boldsymbol{W}^2)]/2 = (\boldsymbol{W} : \boldsymbol{W})/2 = \boldsymbol{w} \cdot \boldsymbol{w}$. Thus, from the characteristic equation, we get the eigenvalues of $\boldsymbol{W}$ as $(0, i\,|\boldsymbol{w}|\,, -i\,|\boldsymbol{w}|)$. By virtue of Eqn. (1.71), the zero eigenvalue obviously corresponds to the eigenvector $\boldsymbol{w}/\,|\boldsymbol{w}|$

1.4 Orthogonal Tensors

A second-order tensor $\boldsymbol{Q}$ is said to be orthogonal if $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$, or, alternatively by Eqn. (1.59), if

$$\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}, \tag{1.73}$$

where $\boldsymbol{I}$ is the identity tensor.

Theorem 1.4.1. *A tensor $\boldsymbol{Q}$ is orthogonal if and only if it has any of the following properties of preserving inner products, lengths and distances:*

$$(\boldsymbol{Qu}, \boldsymbol{Qv}) = (\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{1.74a}$$

$$|\boldsymbol{Qu}| = |\boldsymbol{u}| \quad \forall \boldsymbol{u} \in V, \tag{1.74b}$$

$$|\boldsymbol{Qu} - \boldsymbol{Qv}| = |\boldsymbol{u} - \boldsymbol{v}| \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{1.74c}$$

Proof. Assuming that $\boldsymbol{Q}$ is orthogonal, Eqn. (1.74a) follows since

$$(\boldsymbol{Qu}, \boldsymbol{Qv}) = (\boldsymbol{Q}^T\boldsymbol{Qu}, \boldsymbol{v}) = (\boldsymbol{Iu}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V.$$

Conversely, if Eqn. (1.74a) holds, then

$$0 = (\boldsymbol{Qu}, \boldsymbol{Qv}) - (\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{Q}^T\boldsymbol{Qv}) - (\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, (\boldsymbol{Q}^T\boldsymbol{Q} - \boldsymbol{I})\boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V,$$

which implies that $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$ (by Eqn. (1.21)), and hence $\boldsymbol{Q}$ is orthogonal.

By choosing $\boldsymbol{v} = \boldsymbol{u}$ in Eqn. (1.74a), we get Eqn. (1.74b). Conversely, if Eqn. (1.74b) holds, i.e., if $(\boldsymbol{Qu}, \boldsymbol{Qu}) = (\boldsymbol{u}, \boldsymbol{u})$ for all $\boldsymbol{u} \in V$, then

$$((\boldsymbol{Q}^T\boldsymbol{Q} - \boldsymbol{I})\boldsymbol{u}, \boldsymbol{u}) = 0 \quad \forall \boldsymbol{u} \in V,$$

which, by virtue of Theorem 1.5.3, leads us to the conclusion that $\boldsymbol{Q} \in \text{Orth}$.

By replacing $\boldsymbol{u}$ by $(\boldsymbol{u} - \boldsymbol{v})$ in Eqn. (1.74b), we obtain Eqn. (1.74c), and, conversely, by setting $\boldsymbol{v}$ to zero in Eqn. (1.74c), we get Eqn. (1.74b). □

As a corollary of the above results, it follows that the 'angle' between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, defined by $\theta := \cos^{-1}(\boldsymbol{u} \cdot \boldsymbol{v})/(|\boldsymbol{u}|\,|\boldsymbol{v}|)$, is also preserved. Thus, physically speaking, multiplying the position vectors of all points in a domain by $\boldsymbol{Q}$ corresponds to rigid body rotation of the domain about the origin.

From Eqns. (1.53b), (1.53d) and (1.73), we have $\det \boldsymbol{Q} = \pm 1$. Orthogonal tensors with determinant +1 are said to be proper orthogonal or rotations (henceforth, this set is denoted by Orth^+). For $\boldsymbol{Q} \in \text{Orth}^+$, using Eqn. (1.58), we have

$$\textbf{cof}\, \boldsymbol{Q} = (\det \boldsymbol{Q})\boldsymbol{Q}^{-T} = \boldsymbol{Q}, \tag{1.75}$$

so that by Eqn. (1.44),

$$\boldsymbol{Q}(\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{Qu}) \times (\boldsymbol{Qv}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{1.76}$$

A characterization of a rotation is as follows:

Theorem 1.4.2. *Let* $\{\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \bar{\boldsymbol{e}}_3\}$ *and* $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \boldsymbol{e}_3^*\}$ *be two orthonormal bases. Then*

$$\boldsymbol{Q} = \bar{\boldsymbol{e}}_1 \otimes \boldsymbol{e}_1^* + \bar{\boldsymbol{e}}_2 \otimes \boldsymbol{e}_2^* + \bar{\boldsymbol{e}}_3 \otimes \boldsymbol{e}_3^*,$$

is a proper orthogonal tensor.

Proof. If $\{\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \bar{\boldsymbol{e}}_3\}$ and $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \boldsymbol{e}_3^*\}$ are two orthonormal bases, then

$$\begin{aligned}\boldsymbol{Q}\boldsymbol{Q}^T &= [\bar{\boldsymbol{e}}_1 \otimes \boldsymbol{e}_1^* + \bar{\boldsymbol{e}}_2 \otimes \boldsymbol{e}_2^* + \bar{\boldsymbol{e}}_3 \otimes \boldsymbol{e}_3^*]\,[\boldsymbol{e}_1^* \otimes \bar{\boldsymbol{e}}_1 + \boldsymbol{e}_2^* \otimes \bar{\boldsymbol{e}}_2 + \boldsymbol{e}_3^* \otimes \bar{\boldsymbol{e}}_3] \\ &= [\bar{\boldsymbol{e}}_1 \otimes \bar{\boldsymbol{e}}_1 + \bar{\boldsymbol{e}}_2 \otimes \bar{\boldsymbol{e}}_2 + \bar{\boldsymbol{e}}_3 \otimes \bar{\boldsymbol{e}}_3] && \text{(by Eqn. (1.38))} \\ &= \boldsymbol{I}, && \text{(by Eqn. (1.35))}\end{aligned}$$

It can be shown that $\det \boldsymbol{Q} = 1$. □

If $\{\boldsymbol{e}_i\}$ and $\{\bar{\boldsymbol{e}}_i\}$ are two sets of orthonormal basis vectors, then they are related as

$$\bar{\boldsymbol{e}}_i = \boldsymbol{Q}^T \boldsymbol{e}_i, \quad i = 1, 2, 3, \tag{1.77}$$

where $\boldsymbol{Q} = \boldsymbol{e}_k \otimes \bar{\boldsymbol{e}}_k$ is a proper orthogonal tensor by virtue of Theorem 1.4.2. The components of $\boldsymbol{Q}$ with respect to the $\{\boldsymbol{e}_i\}$ basis are given by $Q_{ij} = \boldsymbol{e}_i \cdot (\boldsymbol{e}_k \otimes \bar{\boldsymbol{e}}_k)\boldsymbol{e}_j = \delta_{ik}\bar{\boldsymbol{e}}_k \cdot \boldsymbol{e}_j = \bar{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j$. Thus, if $\bar{\boldsymbol{e}}$ and $\boldsymbol{e}$ are two unit vectors, we can always find $\boldsymbol{Q} \in \text{Orth}^+$ (not necessarily unique), which rotates $\bar{\boldsymbol{e}}$ to $\boldsymbol{e}$, i.e., $\boldsymbol{e} = \boldsymbol{Q}\bar{\boldsymbol{e}}$. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors. Since $\boldsymbol{u}/\left|\boldsymbol{u}\right|$ and $\boldsymbol{v}/\left|\boldsymbol{v}\right|$ are unit vectors, there exists $\boldsymbol{Q} \in \text{Orth}^+$ such that

$$\frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \boldsymbol{Q}\left(\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\right).$$

Thus, if $\boldsymbol{u}$ and $\boldsymbol{v}$ have the same magnitude, i.e., if $|\boldsymbol{u}| = |\boldsymbol{v}|$, then there exists $\boldsymbol{Q} \in \text{Orth}^+$ such that $\boldsymbol{u} = \boldsymbol{Q}\boldsymbol{v}$.

We now study the transformation laws for the components of tensors under an orthogonal transformation of the basis vectors. Let $\boldsymbol{e}_i$ and $\bar{\boldsymbol{e}}_i$ represent the original and new orthonormal basis vectors, and let $\boldsymbol{Q}$ be the proper orthogonal tensor in Eqn. (1.77). From Eqn. (1.6), we have

$$\bar{\boldsymbol{e}}_i = (\bar{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j)\boldsymbol{e}_j = Q_{ij}\boldsymbol{e}_j, \tag{1.78a}$$

$$\boldsymbol{e}_i = (\boldsymbol{e}_i \cdot \bar{\boldsymbol{e}}_j)\bar{\boldsymbol{e}}_j = Q_{ji}\bar{\boldsymbol{e}}_j. \tag{1.78b}$$

Using Eqn. (1.5) and Eqn. (1.78a), we get the transformation law for the components of a vector as

$$\bar{v}_i = \boldsymbol{v} \cdot \bar{\boldsymbol{e}}_i = \boldsymbol{v} \cdot (Q_{ij}\boldsymbol{e}_j) = Q_{ij}\boldsymbol{v} \cdot \boldsymbol{e}_j = Q_{ij}v_j. \tag{1.79}$$

In a similar fashion, using Eqn. (1.25), Eqn. (1.78a), and the fact that a tensor is a linear transformation, we get the transformation law for the components of a second-order tensor as

$$\bar{T}_{ij} = \bar{\boldsymbol{e}}_i \cdot \boldsymbol{T}\bar{\boldsymbol{e}}_j = Q_{im}Q_{jn}T_{mn}. \tag{1.80}$$

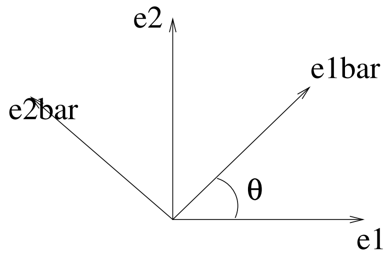

Fig. 1.1: Example of a coordinate system obtained from an existing one by a rotation about the 3-axis.

Conversely, if the components of a matrix transform according to Eqn. (1.80), then they all generate the same tensor. To see this, let $\bar{\boldsymbol{T}} = \bar{T}_{ij}\bar{\boldsymbol{e}}_i \otimes \bar{\boldsymbol{e}}_j$ and $\boldsymbol{T} = T_{mn}\boldsymbol{e}_m \otimes \boldsymbol{e}_n$. Then

$$\begin{aligned} \bar{\boldsymbol{T}} &= \bar{T}_{ij}\bar{\boldsymbol{e}}_i \otimes \bar{\boldsymbol{e}}_j \\ &= Q_{im}Q_{jn}T_{mn}\bar{\boldsymbol{e}}_i \otimes \bar{\boldsymbol{e}}_j \\ &= T_{mn}(Q_{im}\bar{\boldsymbol{e}}_i) \otimes (Q_{jn}\bar{\boldsymbol{e}}_j) \\ &= T_{mn}\boldsymbol{e}_m \otimes \boldsymbol{e}_n \quad \text{(by Eqn. (1.78b))} \\ &= \boldsymbol{T}. \end{aligned}$$

We can write Eqns. (1.79) and (1.80) as

$$[\bar{\boldsymbol{v}}] = \boldsymbol{Q}[\boldsymbol{v}], \tag{1.81}$$

$$[\bar{\boldsymbol{T}}] = \boldsymbol{Q}[\boldsymbol{T}]\boldsymbol{Q}^T. \tag{1.82}$$

where $[\bar{\boldsymbol{v}}]$ and $[\bar{\boldsymbol{T}}]$ represent the components of the vector $\boldsymbol{v}$ and tensor $\boldsymbol{T}$, respectively, with respect to the $\bar{\boldsymbol{e}}_i$ coordinate system. Using the orthogonality property of $\boldsymbol{Q}$, we can write the reverse transformations as

$$[\boldsymbol{v}] = \boldsymbol{Q}^T[\bar{\boldsymbol{v}}], \tag{1.83}$$

$$[\boldsymbol{T}] = \boldsymbol{Q}^T[\bar{\boldsymbol{T}}]\boldsymbol{Q}. \tag{1.84}$$

As an example, the $\boldsymbol{Q}$ matrix for the configuration shown in Fig. 1.1 is

$$\boldsymbol{Q} = \begin{bmatrix} \bar{\boldsymbol{e}}_1 \\ \bar{\boldsymbol{e}}_2 \\ \bar{\boldsymbol{e}}_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From Eqn. (1.82), it follows that

$$\det([\bar{\boldsymbol{T}}] - \lambda\boldsymbol{I}) = \det(\boldsymbol{Q}[\boldsymbol{T}]\boldsymbol{Q}^T - \lambda\boldsymbol{I})$$

$$
\begin{aligned}
&= \det(\boldsymbol{Q}[\boldsymbol{T}]\boldsymbol{Q}^T - \lambda\boldsymbol{Q}\boldsymbol{Q}^T)\\
&= (\det\boldsymbol{Q})\det([\boldsymbol{T}] - \lambda\boldsymbol{I})(\det\boldsymbol{Q}^T)\\
&= \det(\boldsymbol{Q}\boldsymbol{Q}^T)\det([\boldsymbol{T}] - \lambda\boldsymbol{I})\\
&= \det([\boldsymbol{T}] - \lambda\boldsymbol{I}),
\end{aligned}
$$

which shows that the characteristic equation, and hence the principal invariants I_1, I_2 and I_3 of $[\bar{\boldsymbol{T}}]$ and $[\boldsymbol{T}]$ are the same. Thus, although the component matrices $[\bar{\boldsymbol{T}}]$ and $[\boldsymbol{T}]$ are different, their trace, second invariant and determinant are the same, and hence the term *invariant* is used for them.

The only real eigenvalues of $\boldsymbol{Q} \in \text{Orth}$ can be either $+1$ or -1, since if λ and $\boldsymbol{n}$ denote the eigenvalue and eigenvector of $\boldsymbol{Q}$, i.e., $\boldsymbol{Q}\boldsymbol{n} = \lambda\boldsymbol{n}$, then

$$
(\boldsymbol{n}, \boldsymbol{n}) = (\boldsymbol{Q}\boldsymbol{n}, \boldsymbol{Q}\boldsymbol{n}) = \lambda^2(\boldsymbol{n}, \boldsymbol{n}),
$$

which implies that $(\boldsymbol{n}, \boldsymbol{n})(\lambda^2 - 1) = 0$. If λ and $\boldsymbol{n}$ are real, then $(\boldsymbol{n}, \boldsymbol{n}) \neq 0$ and $\lambda = \pm 1$, while if λ is complex, then $(\boldsymbol{n}, \boldsymbol{n}) = 0$. Let $\hat{\lambda}$ and $\hat{\boldsymbol{n}}$ denote the complex conjugates of λ and $\boldsymbol{n}$, respectively. To see that the complex eigenvalues have a magnitude of unity observe that $\lambda\hat{\lambda}(\hat{\boldsymbol{n}}, \boldsymbol{n}) = (\boldsymbol{Q}\hat{\boldsymbol{n}}, \boldsymbol{Q}\boldsymbol{n}) = (\hat{\boldsymbol{n}}, \boldsymbol{n})$, which implies that $\lambda\hat{\lambda} = 1$ since $(\hat{\boldsymbol{n}}, \boldsymbol{n}) \neq 0$.

If $\boldsymbol{R} \neq \boldsymbol{I}$ is a rotation, then the set of all vectors $\boldsymbol{e}$ such that

$$
\boldsymbol{R}\boldsymbol{e} = \boldsymbol{e} \tag{1.85}
$$

forms a one-dimensional subspace of V called the *axis* of $\boldsymbol{R}$. To prove that such a vector always exists, we first show that $+1$ is always an eigenvalue of $\boldsymbol{R}$. Since $\det\boldsymbol{R} = 1$,

$$
\begin{aligned}
\det(\boldsymbol{R} - \boldsymbol{I}) &= \det(\boldsymbol{R} - \boldsymbol{R}\boldsymbol{R}^T) = (\det\boldsymbol{R})\det(\boldsymbol{I} - \boldsymbol{R}^T) = \det(\boldsymbol{I} - \boldsymbol{R}^T)^T\\
&= \det(\boldsymbol{I} - \boldsymbol{R}) = -\det(\boldsymbol{R} - \boldsymbol{I}),
\end{aligned}
$$

which implies that $\det(\boldsymbol{R} - \boldsymbol{I}) = 0$, or that $+1$ is an eigenvalue. If $\boldsymbol{e}$ is the eigenvector corresponding to the eigenvalue $+1$, then $\boldsymbol{R}\boldsymbol{e} = \boldsymbol{e}$.

Conversely, given a vector $\boldsymbol{w}$, there exists a proper orthogonal tensor $\boldsymbol{R}$, such that $\boldsymbol{R}\boldsymbol{w} = \boldsymbol{w}$. To see this, consider the family of tensors

$$
\boldsymbol{R}(\boldsymbol{w}, \alpha) = \boldsymbol{I} + \frac{1}{|\boldsymbol{w}|}\sin\alpha\,\boldsymbol{W} + \frac{1}{|\boldsymbol{w}|^2}(1 - \cos\alpha)\boldsymbol{W}^2, \tag{1.86}
$$

where $\boldsymbol{W}$ is the skew-symmetric tensor with $\boldsymbol{w}$ as its axial vector, i.e., $\boldsymbol{W}\boldsymbol{w} = \boldsymbol{0}$. Using the Cayley–Hamilton theorem, we have $\boldsymbol{W}^3 = -|\boldsymbol{w}|^2\boldsymbol{W}$, from which it follows that $\boldsymbol{W}^4 = -|\boldsymbol{w}|^2\boldsymbol{W}^2$. Using this result, we get

$$
\boldsymbol{R}^T\boldsymbol{R} = \left[\boldsymbol{I} - \frac{\sin\alpha}{|\boldsymbol{w}|}\boldsymbol{W} + \frac{(1 - \cos\alpha)}{|\boldsymbol{w}|^2}\boldsymbol{W}^2\right]\left[\boldsymbol{I} + \frac{\sin\alpha}{|\boldsymbol{w}|}\boldsymbol{W} + \frac{(1 - \cos\alpha)}{|\boldsymbol{w}|^2}\boldsymbol{W}^2\right] = \boldsymbol{I}.
$$

Since $\boldsymbol{R}$ has now been shown to be orthogonal, $\det \boldsymbol{R} = \pm 1$. However, since $\det[\boldsymbol{R}(\boldsymbol{w}, 0)] = \det \boldsymbol{I} = 1$, by continuity, we have $\det[\boldsymbol{R}(\boldsymbol{w}, \alpha)] = 1$ for any α. Thus, $\boldsymbol{R}$ is a proper orthogonal tensor that satisfies $\boldsymbol{R}\boldsymbol{w} = \boldsymbol{w}$. It is easily seen that $\boldsymbol{R}\boldsymbol{w} = \boldsymbol{w}$. Essentially, $\boldsymbol{R}$ rotates any vector in the plane perpendicular to $\boldsymbol{w}$ through an angle α.

1.5 Symmetric Tensors

In this section, we examine some properties of symmetric second-order tensors. We first discuss the properties of the principal values (eigenvalues) and principal directions (eigenvectors) of a symmetric second-order tensor.

1.5.1 Principal values and principal directions

We have the following result:

Theorem 1.5.1. *Every symmetric tensor $\boldsymbol{S}$ has at least one principal frame, i.e., a right-handed triplet of orthogonal principal directions, and at most three distinct principal values. The principal values are always real. For the principal directions three possibilities exist:*

- *If all the three principal values are distinct, the principal axes are unique (modulo sign reversal).*
- *If two eigenvalues are equal, then there is one unique principal direction, and the remaining two principal directions can be chosen arbitrarily in the plane perpendicular to the first one, and mutually perpendicular to each other.*
- *If all three eigenvalues are the same, then every right-handed frame is a principal frame, and $\boldsymbol{S}$ is of the form $\boldsymbol{S} = \lambda \boldsymbol{I}$.*

The components of the tensor in the principal frame are

$$\boldsymbol{S}^* = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{1.87}$$

Proof. We seek λ and $\boldsymbol{n}$ such that

$$(\boldsymbol{S} - \lambda \boldsymbol{I})\boldsymbol{n} = \boldsymbol{0}. \tag{1.88}$$

But this is nothing but an eigenvalue problem. For a nontrivial solution, we need to satisfy the condition that

$$\det(\boldsymbol{S} - \lambda \boldsymbol{I}) = 0,$$

or, by Eqn. (1.63),

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0, \tag{1.89}$$

where I_1, I_2 and I_3 are the principal invariants of $\boldsymbol{S}$.

We now show that the principal values given by the three roots of the cubic equation Eqn. (1.89) are real. Suppose that two roots, and hence the eigenvectors associated with them, are complex. Denoting the complex conjugates of λ and $\boldsymbol{n}$ by $\hat{\lambda}$ and $\hat{\boldsymbol{n}}$, we have

$$\boldsymbol{S}\boldsymbol{n} = \lambda\boldsymbol{n}, \tag{1.90a}$$

$$\boldsymbol{S}\hat{\boldsymbol{n}} = \hat{\lambda}\hat{\boldsymbol{n}}, \tag{1.90b}$$

where Eqn. (1.90b) is obtained by taking the complex conjugate of Eqn. (1.90a) ($\boldsymbol{S}$ being a real matrix is not affected). Taking the dot product of both sides of Eqn. (1.90a) with $\hat{\boldsymbol{n}}$, and of both sides of Eqn. (1.90b) with $\boldsymbol{n}$, we get

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{S}\boldsymbol{n} = \lambda\boldsymbol{n} \cdot \hat{\boldsymbol{n}}, \tag{1.91}$$

$$\boldsymbol{n} \cdot \boldsymbol{S}\hat{\boldsymbol{n}} = \hat{\lambda}\hat{\boldsymbol{n}} \cdot \boldsymbol{n}. \tag{1.92}$$

Using the definition of a transpose of a tensor, and subtracting the second relation from the first, we get

$$\boldsymbol{S}^T\hat{\boldsymbol{n}} \cdot \boldsymbol{n} - \boldsymbol{n} \cdot \boldsymbol{S}\hat{\boldsymbol{n}} = (\lambda - \hat{\lambda})\boldsymbol{n} \cdot \hat{\boldsymbol{n}}. \tag{1.93}$$

Since $\boldsymbol{S}$ is symmetric, $\boldsymbol{S}^T = \boldsymbol{S}$, and we have

$$(\lambda - \hat{\lambda})\boldsymbol{n} \cdot \hat{\boldsymbol{n}} = 0.$$

Since $\boldsymbol{n} \cdot \hat{\boldsymbol{n}} \neq 0$, $\lambda = \hat{\lambda}$, and hence the eigenvalues are real.

The principal directions $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$, corresponding to distinct eigenvalues λ_1, λ_2 and λ_3, are mutually orthogonal and unique (modulo sign reversal). We now prove this. Taking the dot product of

$$\boldsymbol{S}\boldsymbol{n}_1 = \lambda_1\boldsymbol{n}_1, \tag{1.94}$$

$$\boldsymbol{S}\boldsymbol{n}_2 = \lambda_2\boldsymbol{n}_2, \tag{1.95}$$

with $\boldsymbol{n}_2$ and $\boldsymbol{n}_1$, respectively, and subtracting, we get

$$0 = (\lambda_1 - \lambda_2)\boldsymbol{n}_1 \cdot \boldsymbol{n}_2,$$

where we have used the fact that $\boldsymbol{S}$ being symmetric, $\boldsymbol{n}_2 \cdot \boldsymbol{S}\boldsymbol{n}_1 - \boldsymbol{n}_1 \cdot \boldsymbol{S}\boldsymbol{n}_2 = 0$. Thus, since we assumed that $\lambda_1 \neq \lambda_2$, we get $\boldsymbol{n}_1 \perp \boldsymbol{n}_2$. Similarly, we have $\boldsymbol{n}_2 \perp \boldsymbol{n}_3$ and $\boldsymbol{n}_1 \perp \boldsymbol{n}_3$. If $\boldsymbol{n}_1$

satisfies $\boldsymbol{S}\boldsymbol{n}_1 = \lambda_1 \boldsymbol{n}_1$, then we see that $-\boldsymbol{n}_1$ also satisfies the same equation. This is the only other choice possible that satisfies $\boldsymbol{S}\boldsymbol{n}_1 = \lambda_1 \boldsymbol{n}_1$. To see this, let $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ be another set of mutually perpendicular eigenvectors corresponding to the distinct eigenvalues λ_1, λ_2 and λ_3. Then $\boldsymbol{r}_1$ has to be perpendicular to not only $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$, but to $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ as well. Similar comments apply to $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$. This is only possible when $\boldsymbol{r}_1 = \pm\boldsymbol{n}_1$, $\boldsymbol{r}_2 = \pm\boldsymbol{n}_2$ and $\boldsymbol{r}_3 = \pm\boldsymbol{n}_3$. Thus, the principal axes are unique modulo sign reversal.

To prove that the components of $\boldsymbol{S}$ in the principal frame are given by Eqn. (1.87), assume that $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$ have been normalized to unit length, and then let $\boldsymbol{e}_1^* = \boldsymbol{n}_1$, $\boldsymbol{e}_2^* = \boldsymbol{n}_2$ and $\boldsymbol{e}_3^* = \boldsymbol{n}_3$. Using Eqn. (1.25), and taking into account the orthonormality of $\boldsymbol{e}_1^*$ and $\boldsymbol{e}_2^*$, the components S_{11}^* and S_{12}^* are given by

$$S_{11}^* = \boldsymbol{e}_1^* \cdot \boldsymbol{S}\boldsymbol{e}_1^* = \boldsymbol{e}_1^* \cdot (\lambda_1 \boldsymbol{e}_1^*) = \lambda_1,$$
$$S_{12}^* = \boldsymbol{e}_1^* \cdot \boldsymbol{S}\boldsymbol{e}_2^* = \boldsymbol{e}_1^* \cdot (\lambda_2 \boldsymbol{e}_2^*) = 0.$$

Similarly, on computing the other components, we see that the matrix representation of $\boldsymbol{S}$ with respect to $\boldsymbol{e}^*$ is given by Eqn. (1.87).

If there are two repeated roots, say, $\lambda_2 = \lambda_3$, and the third root $\lambda_1 \neq \lambda_2$, then let $\boldsymbol{e}_1^*$ coincide with $\boldsymbol{n}_1$, so that $\boldsymbol{S}\boldsymbol{e}_1^* = \lambda_1 \boldsymbol{e}_1^*$. Choose $\boldsymbol{e}_2^*$ and $\boldsymbol{e}_3^*$ such that $\boldsymbol{e}_1^*$-$\boldsymbol{e}_3^*$ form a right-handed orthogonal coordinate system. The components of $\boldsymbol{S}$ with respect to this coordinate system are

$$\boldsymbol{S}^* = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & S_{22}^* & S_{23}^* \\ 0 & S_{23}^* & S_{33}^* \end{bmatrix}. \tag{1.96}$$

By Eqn. (1.64), we have

$$\begin{gathered} S_{22}^* + S_{33}^* = 2\lambda_2, \\ \lambda_1 S_{22}^* + (S_{22}^* S_{33}^* - (S_{23}^*)^2) + \lambda_1 S_{33}^* = 2\lambda_1\lambda_2 + \lambda_2^2, \\ \lambda_1 \left[S_{22}^* S_{33}^* - (S_{23}^*)^2\right] = \lambda_1 \lambda_2^2. \end{gathered} \tag{1.97}$$

Substituting for λ_2 from the first equation into the second, we get

$$(S_{22}^* - S_{33}^*)^2 = -4(S_{23}^*)^2.$$

Since the components of $\boldsymbol{S}$ are real, the above equation implies that $S_{23}^* = 0$ and $\lambda_2 = S_{22}^* = S_{33}^*$. This shows that Eqn. (1.96) reduces to Eqn. (1.87), and that $\boldsymbol{S}\boldsymbol{e}_2^* = S_{12}^* \boldsymbol{e}_1^* + S_{22}^* \boldsymbol{e}_2^* + S_{32}^* \boldsymbol{e}_3^* = \lambda_2 \boldsymbol{e}_2^*$ and $\boldsymbol{S}\boldsymbol{e}_3^* = \lambda_2 \boldsymbol{e}_3^*$ (thus, $\boldsymbol{e}_2^*$ and $\boldsymbol{e}_3^*$ are eigenvectors corresponding to the eigenvalue λ_2). However, in this case the choice of the principal frame $\boldsymbol{e}^*$ is not unique, since any vector lying in the plane of $\boldsymbol{e}_2^*$ and $\boldsymbol{e}_3^*$, given by $\boldsymbol{n}^* = c_1 \boldsymbol{e}_2^* + c_2 \boldsymbol{e}_3^*$ where c_1 and c_2 are arbitrary constants, is also an eigenvector. The choice of $\boldsymbol{e}_1^*$ is unique (modulo sign reversal), since it has to be perpendicular to $\boldsymbol{e}_2^*$ and $\boldsymbol{e}_3^*$. Though the choice of $\boldsymbol{e}_2^*$ and $\boldsymbol{e}_3^*$ is not unique, we can choose $\boldsymbol{e}_2^*$ and $\boldsymbol{e}_3^*$ arbitrarily in the plane perpendicular to $\boldsymbol{e}_1^*$, and such that $\boldsymbol{e}_2^* \perp \boldsymbol{e}_3^*$.

Finally, if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then the tensor $\boldsymbol{S}$ is of the form $\boldsymbol{S} = \lambda \boldsymbol{I}$. To show this choose $\boldsymbol{e}_1^*$–$\boldsymbol{e}_3^*$, and follow a procedure analogous to that in the previous case. We now get $S^*_{22} = S^*_{33} = \lambda$ and $S^*_{23} = 0$, so that $[\boldsymbol{S}^*] = \lambda \boldsymbol{I}$. Using the transformation law for second-order tensors, we have

$$[\boldsymbol{S}] = \boldsymbol{Q}^T [\boldsymbol{S}^*] \boldsymbol{Q} = \lambda \boldsymbol{Q}^T \boldsymbol{Q} = \lambda \boldsymbol{I}.$$

Thus, any arbitrary vector $\boldsymbol{n}$ is a solution of $\boldsymbol{S}\boldsymbol{n} = \lambda \boldsymbol{n}$, and hence every right-handed frame is a principal frame. □

As a result of Eqn. (1.87), we can write

$$\boldsymbol{S} = \sum_{i=1}^{3} \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^* = \lambda_1 \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2 \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_3 \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*, \tag{1.98}$$

which is called as the *spectral resolution* of $\boldsymbol{S}$. The spectral resolution of $\boldsymbol{S}$ is unique since

- If all the eigenvalues are distinct, then the eigenvectors are unique, and consequently the representation given by Eqn. (1.98) is unique.

- If two eigenvalues are repeated, then, by virtue of Eqn. (1.35), Eqn. (1.98) reduces to

$$\begin{aligned} \boldsymbol{S} &= \lambda_1 \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2 \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_2 \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^* \\ &= \lambda_1 \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2 (\boldsymbol{I} - \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^*), \end{aligned} \tag{1.99}$$

  from which the asserted uniqueness follows, since $\boldsymbol{e}_1^*$ is unique.

- If all the eigenvalues are the same then $\boldsymbol{S} = \lambda \boldsymbol{I}$.

The Cayley–Hamilton theorem (Theorem 1.2.3) applied to $\boldsymbol{S} \in \text{Sym}$ yields

$$\boldsymbol{S}^3 - I_1 \boldsymbol{S}^2 + I_2 \boldsymbol{S} - I_3 \boldsymbol{I} = \boldsymbol{0}. \tag{1.100}$$

We have already proved this result for any arbitrary tensor. However, the following simpler proof can be given for symmetric tensors. Using Eqn. (1.38), the spectral resolutions of $\boldsymbol{S}$, $\boldsymbol{S}^2$ and $\boldsymbol{S}^3$ are

$$\begin{aligned} \boldsymbol{S} &= \lambda_1 \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2 \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_3 \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*, \\ \boldsymbol{S}^2 &= \lambda_1^2 \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2^2 \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_3^2 \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*, \\ \boldsymbol{S}^3 &= \lambda_1^3 \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2^3 \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_3^3 \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*. \end{aligned} \tag{1.101}$$

Substituting these expressions into the left-hand side of Eqn. (1.100), we get

$$\begin{aligned} \text{LHS} = (\lambda_1^3 - I_1 \lambda_1^2 + I_2 \lambda_1 - I_3)(\boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^*) + (\lambda_2^3 - I_1 \lambda_2^2 + I_2 \lambda_2 - I_3)(\boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^*) \\ + (\lambda_3^3 - I_1 \lambda_3^2 + I_2 \lambda_3 - I_3)(\boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*) = \boldsymbol{0}, \end{aligned}$$

since $\lambda_i^3 - I_1 \lambda_i^2 + I_2 \lambda_i - I_3 = 0$ for $i = 1, 2, 3$.

1.5.2 Positive definite tensors and the polar decomposition

A second-order symmetric tensor $\boldsymbol{S}$ is positive definite if

$$(\boldsymbol{u}, \boldsymbol{S}\boldsymbol{u}) \geq 0 \quad \forall \boldsymbol{u} \in V \text{ with } (\boldsymbol{u}, \boldsymbol{S}\boldsymbol{u}) = 0 \text{ if and only if } \boldsymbol{u} = \boldsymbol{0}.$$

We denote the set of symmetric, positive definite tensors by Psym. Since by virtue of Eqn. (1.30), all tensors $\boldsymbol{T}$ can be decomposed into a symmetric part $\boldsymbol{T}_s$, and a skew-symmetric part $\boldsymbol{T}_{ss}$, we have

$$\begin{aligned}(\boldsymbol{u}, \boldsymbol{T}\boldsymbol{u}) &= (\boldsymbol{u}, \boldsymbol{T}_s\boldsymbol{u}) + (\boldsymbol{u}, \boldsymbol{T}_{ss}\boldsymbol{u}),\\ &= (\boldsymbol{u}, \boldsymbol{T}_s\boldsymbol{u}),\end{aligned}$$

because $(\boldsymbol{u}, \boldsymbol{T}_{ss}\boldsymbol{u}) = 0$ by Eqn. (1.70). Thus, the positive definiteness of a tensor is decided by the positive definiteness of its symmetric part. In Theorem 1.5.2, we show that a symmetric tensor is positive definite if and only if its eigenvalues are positive. Although the eigenvalues of the *symmetric part* of $\boldsymbol{T}$ should be positive in order for $\boldsymbol{T}$ to be positive definite, positiveness of the eigenvalues of $\boldsymbol{T}$ itself does not ensure its positive definiteness as the following counterexample shows. If $\boldsymbol{T} = \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix}$, then $\boldsymbol{T}$ is not positive definite since $\boldsymbol{u} \cdot \boldsymbol{T}\boldsymbol{u} < 0$ for $\boldsymbol{u} = (1, 1)$, but the eigenvalues of $\boldsymbol{T}$ are $(1, 1)$. Conversely, if $\boldsymbol{T}$ is positive definite, then by choosing $\boldsymbol{u}$ to be the real eigenvectors $\boldsymbol{n}$ of $\boldsymbol{T}$, it follows that its real eigenvalues $\lambda = (\boldsymbol{n} \cdot \boldsymbol{T}\boldsymbol{n})$ are positive.

Theorem 1.5.2. *Let $\boldsymbol{S} \in$ Sym. Then the following are equivalent:*

1. *$\boldsymbol{S}$ is positive definite.*
2. *The principal values of $\boldsymbol{S}$ are strictly positive.*
3. *The principal invariants of $\boldsymbol{S}$ are strictly positive.*

Proof. We first prove the equivalence of (1) and (2). Suppose $\boldsymbol{S}$ is positive definite. If λ and $\boldsymbol{n}$ denote the principal values and principal directions, respectively, of $\boldsymbol{S}$, then $\boldsymbol{S}\boldsymbol{n} = \lambda\boldsymbol{n}$, which implies that $\lambda = (\boldsymbol{n}, \boldsymbol{S}\boldsymbol{n}) > 0$ since $\boldsymbol{n} \neq \boldsymbol{0}$.

Conversely, suppose that the principal values of $\boldsymbol{S}$ are greater than 0. Assuming that $\boldsymbol{e}_1^*$, $\boldsymbol{e}_2^*$ and $\boldsymbol{e}_3^*$ denote the principal axes, the representation of $\boldsymbol{S}$ in the principal coordinate frame is (see Eqn. (1.98))

$$\boldsymbol{S} = \lambda_1 \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2 \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_3 \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*.$$

Then

$$\begin{aligned}\boldsymbol{S}\boldsymbol{u} &= (\lambda_1 \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2 \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_3 \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*)\boldsymbol{u}\\ &= \lambda_1(\boldsymbol{e}_1^* \cdot \boldsymbol{u})\boldsymbol{e}_1^* + \lambda_2(\boldsymbol{e}_2^* \cdot \boldsymbol{u})\boldsymbol{e}_2^* + \lambda_3(\boldsymbol{e}_3^* \cdot \boldsymbol{u})\boldsymbol{e}_3^*\\ &= \lambda_1 u_1^* \boldsymbol{e}_1^* + \lambda_2 u_2^* \boldsymbol{e}_2^* + \lambda_3 u_3^* \boldsymbol{e}_3^*,\end{aligned}$$

and

$$(\boldsymbol{u}, \boldsymbol{S}\boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{S}\boldsymbol{u} = \lambda_1 (u_1^*)^2 + \lambda_2 (u_2^*)^2 + \lambda_3 (u_3^*)^2, \tag{1.102}$$

which is greater than or equal to zero since $\lambda_i > 0$. Suppose that $(\boldsymbol{u}, \boldsymbol{S}\boldsymbol{u}) = 0$. Then by Eqn. (1.102), $u_i^* = 0$, which implies that $\boldsymbol{u} = \boldsymbol{0}$. Thus, $\boldsymbol{S}$ is a positive definite tensor.

To prove the equivalence of (2) and (3), note that, by Eqn. (1.64), if all the principal values are strictly positive, then the principal invariants are also strictly positive. Conversely, if all the principal invariants are positive, then $I_3 = \lambda_1\lambda_2\lambda_3$, is positive, so that all the λ_i are nonzero in addition to being real. Each λ_i has to satisfy the characteristic equation

$$\lambda_i^3 - I_1\lambda_i^2 + I_2\lambda_i - I_3 = 0, \quad i = 1, 2, 3.$$

If λ_i is negative, then, since I_1, I_2, I_3 are positive, the left-hand side of the above equation is negative, and hence the above equation cannot be satisfied. We have already mentioned that λ_i cannot be zero. Hence, each λ_i has to be positive. □

Theorem 1.5.3. *For $\boldsymbol{S} \in Sym$,*

$$(\boldsymbol{u}, \boldsymbol{S}\boldsymbol{u}) = 0 \quad \forall \boldsymbol{u} \in V,$$

if and only if $\boldsymbol{S} = \boldsymbol{0}$.

Proof. If $\boldsymbol{S} = \boldsymbol{0}$, then obviously, $(\boldsymbol{u}, \boldsymbol{S}\boldsymbol{u}) = 0$. Conversely, using the fact that $\boldsymbol{S} = \sum_{i=1}^3 \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*$, we get

$$0 = (\boldsymbol{u}, \boldsymbol{S}\boldsymbol{u}) = \sum_{i=1}^3 \lambda_i (\boldsymbol{u}_i^*)^2 \quad \forall \boldsymbol{u}.$$

Choosing $\boldsymbol{u}$ such that $\boldsymbol{u}_1^* \neq 0$, $\boldsymbol{u}_2^* = \boldsymbol{u}_3^* = 0$, we get $\lambda_1 = 0$. Similarly, we can show that $\lambda_2 = \lambda_3 = 0$, so that $\boldsymbol{S} = \sum_{i=1}^3 \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^* = \boldsymbol{0}$. □

Theorem 1.5.4. *If $\boldsymbol{S} \in Psym$, then there exists a unique $\boldsymbol{H} \in Psym$, such that $\boldsymbol{H}^2 := \boldsymbol{H}\boldsymbol{H} = \boldsymbol{S}$. The tensor $\boldsymbol{H}$ is called the positive definite square root of $\boldsymbol{S}$, and we write $\boldsymbol{H} = \sqrt{\boldsymbol{S}}$.*

Proof. Before we begin the proof, we note that a positive definite, symmetric tensor can have square roots that are not positive definite. For example, $\mathbf{diag}[1, -1, 1]$ is a non-positive definite square root of $\boldsymbol{I}$. Here, we are interested only in those square roots that are positive definite.

Since

$$\boldsymbol{S} = \sum_{i=1}^3 \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*,$$

is positive definite, by Theorem 1.5.2, all $\lambda_i > 0$. Define

$$\boldsymbol{H} := \sum_{i=1}^{3} \sqrt{\lambda_i} \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*. \tag{1.103}$$

Since the $\{\boldsymbol{e}_i^*\}$ are orthonormal, it is easily seen that $\boldsymbol{H}\boldsymbol{H} = \boldsymbol{S}$. Since the eigenvalues of $\boldsymbol{H}$ given by $\sqrt{\lambda_i}$ are all positive, $\boldsymbol{H}$ is positive definite. Thus, we have shown that a positive definite square root tensor of $\boldsymbol{S}$ given by Eqn. (1.103) exists. We now prove uniqueness.

With λ and $\boldsymbol{n}$ denoting the principal value and principal direction of $\boldsymbol{S}$, we have

$$\begin{aligned}\mathbf{0} &= (\boldsymbol{S} - \lambda \boldsymbol{I})\boldsymbol{n} \\ &= (\boldsymbol{H}^2 - \lambda \boldsymbol{I})\boldsymbol{n} \\ &= (\boldsymbol{H} + \sqrt{\lambda}\boldsymbol{I})(\boldsymbol{H} - \sqrt{\lambda}\boldsymbol{I})\boldsymbol{n}.\end{aligned}$$

Calling $(\boldsymbol{H} - \sqrt{\lambda}\boldsymbol{I})\boldsymbol{n} = \bar{\boldsymbol{n}}$, we have

$$(\boldsymbol{H} + \sqrt{\lambda}\boldsymbol{I})\bar{\boldsymbol{n}} = \mathbf{0}.$$

This implies that $\bar{\boldsymbol{n}} = \mathbf{0}$. For, if not, $-\sqrt{\lambda}$ is a principal value of $\boldsymbol{H}$, which contradicts the fact that $\boldsymbol{H}$ is positive definite. Therefore,

$$(\boldsymbol{H} - \sqrt{\lambda}\boldsymbol{I})\boldsymbol{n} = \mathbf{0};$$

i.e., $\boldsymbol{n}$ is also a principal direction of $\boldsymbol{H}$ with associated principal values $\sqrt{\lambda}$. If $\boldsymbol{H} = \sqrt{\boldsymbol{S}}$, then it must have the form given by Eqn. (1.103) (since the spectral decomposition is unique), which establishes its uniqueness. □

Now we prove the polar decomposition theorem.

Theorem 1.5.5 (Polar Decomposition Theorem)**.** *Let $\boldsymbol{F}$ be an invertible tensor. Then, it can be factored in a unique fashion as*

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R},$$

where $\boldsymbol{R}$ is an orthogonal tensor, and $\boldsymbol{U}$, $\boldsymbol{V}$ are symmetric and positive definite tensors. One has

$$\begin{aligned}\boldsymbol{U} &= \sqrt{\boldsymbol{F}^T\boldsymbol{F}} \\ \boldsymbol{V} &= \sqrt{\boldsymbol{F}\boldsymbol{F}^T}.\end{aligned}$$

Proof. The tensor $\boldsymbol{F}^T\boldsymbol{F}$ is obviously symmetric. It is positive definite since

$$(\boldsymbol{u}, \boldsymbol{F}^T\boldsymbol{F}\boldsymbol{u}) = (\boldsymbol{F}\boldsymbol{u}, \boldsymbol{F}\boldsymbol{u}) \geq 0,$$

with equality if and only if $\boldsymbol{u} = \boldsymbol{0}$ ($\boldsymbol{F}\boldsymbol{u} = \boldsymbol{0}$ implies that $\boldsymbol{u} = \boldsymbol{0}$, since $\boldsymbol{F}$ is invertible). Let $\boldsymbol{U} = \sqrt{\boldsymbol{F}^T\boldsymbol{F}}$. $\boldsymbol{U}$ is unique, symmetric and positive definite by Theorem 1.5.4. Define $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$, so that $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$. The tensor $\boldsymbol{R}$ is orthogonal, because

$$\begin{aligned}
\boldsymbol{R}^T\boldsymbol{R} &= (\boldsymbol{F}\boldsymbol{U}^{-1})^T(\boldsymbol{F}\boldsymbol{U}^{-1}) \\
&= \boldsymbol{U}^{-T}\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{U}^{-1} \\
&= \boldsymbol{U}^{-1}(\boldsymbol{F}^T\boldsymbol{F})\boldsymbol{U}^{-1} \quad \text{(since } \boldsymbol{U} \text{ is symmetric)} \\
&= \boldsymbol{U}^{-1}(\boldsymbol{U}\boldsymbol{U})\boldsymbol{U}^{-1} \\
&= \boldsymbol{I}.
\end{aligned}$$

Since $\det \boldsymbol{U} > 0$, we have $\det \boldsymbol{U}^{-1} > 0$. Hence, $\det \boldsymbol{R}$ and $\det \boldsymbol{F}$ have the same sign. Usually, the polar decomposition theorem is applied to the deformation gradient $\boldsymbol{F}$ satisfying $\det \boldsymbol{F} > 0$. In such a case $\det \boldsymbol{R} = 1$, and $\boldsymbol{R}$ is a rotation.

Next, let $\boldsymbol{V} = \boldsymbol{F}\boldsymbol{U}\boldsymbol{F}^{-1} = \boldsymbol{F}\boldsymbol{R}^{-1} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^{-1} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T$. Thus, $\boldsymbol{V}$ is symmetric since $\boldsymbol{U}$ is symmetric. $\boldsymbol{V}$ is positive definite since

$$\begin{aligned}
(\boldsymbol{u}, \boldsymbol{V}\boldsymbol{u}) &= (\boldsymbol{u}, \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T\boldsymbol{u}) \\
&= (\boldsymbol{R}^T\boldsymbol{u}, \boldsymbol{U}\boldsymbol{R}^T\boldsymbol{u}) \\
&\geq 0,
\end{aligned}$$

with equality if and only if $\boldsymbol{u} = \boldsymbol{0}$ (again since $\boldsymbol{R}^T$ is invertible). Note that $\boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{V}\boldsymbol{V} = \boldsymbol{V}^2$, so that $\boldsymbol{V} = \sqrt{\boldsymbol{F}\boldsymbol{F}^T}$.

Finally, to prove the uniqueness of the polar decomposition, we note that since $\boldsymbol{U}$ is unique, $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$ is unique, and hence so is $\boldsymbol{V}$. □

Let $(\lambda_i, \boldsymbol{e}_i^*)$ denote the eigenvalues/eigenvectors of $\boldsymbol{U}$. Then, since $\boldsymbol{V}\boldsymbol{R}\boldsymbol{e}_i^* = \boldsymbol{R}\boldsymbol{U}\boldsymbol{e}_i^* = \lambda_i(\boldsymbol{R}\boldsymbol{e}_i^*)$, the pairs $(\lambda_i, \boldsymbol{f}_i^*)$, where $\boldsymbol{f}_i \equiv \boldsymbol{R}\boldsymbol{e}_i^*$ are the eigenvalues/eigenvectors of $\boldsymbol{V}$. Thus, $\boldsymbol{F}$ and $\boldsymbol{R}$ can be represented as

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{R}\sum_{i=1}^{3} \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^* = \sum_{i=1}^{3} \lambda_i (\boldsymbol{R}\boldsymbol{e}_i^*) \otimes \boldsymbol{e}_i^* = \sum_{i=1}^{3} \lambda_i \boldsymbol{f}_i^* \otimes \boldsymbol{e}_i^*, \tag{1.104a}$$

$$\boldsymbol{R} = \boldsymbol{R}\boldsymbol{I} = \boldsymbol{R}\sum_{i=1}^{3} \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^* = \sum_{i} (\boldsymbol{R}\boldsymbol{e}_i^*) \otimes \boldsymbol{e}_i^* = \sum_{i=1}^{3} \boldsymbol{f}_i^* \otimes \boldsymbol{e}_i^*. \tag{1.104b}$$

1.6 Singular Value Decomposition (SVD)

Once the polar decomposition is known, the singular value decomposition (SVD) can be computed for a matrix of dimension n using Eqn. (1.104) as

$$\begin{aligned} \boldsymbol{F} &= \left(\sum_{i=1}^{n} \boldsymbol{f}_i^* \otimes \boldsymbol{e}_i\right)\left(\sum_{i=1}^{n} \lambda_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i\right)\left(\sum_{i=1}^{n} \boldsymbol{e}_i \otimes \boldsymbol{e}_i^*\right) \\ &= \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{Q}^T, \end{aligned} \tag{1.105}$$

where $\{\boldsymbol{e}_i\}$ denotes the canonical basis, and

$$\boldsymbol{\Lambda} = \mathbf{diag}[\lambda_1, \ldots, \lambda_n], \tag{1.106a}$$

$$\boldsymbol{Q} = \sum_{i=1}^{n} \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i = \left[\boldsymbol{e}_1^* \mid \boldsymbol{e}_2^* \mid \ldots \mid \boldsymbol{e}_n^*\right], \tag{1.106b}$$

$$\boldsymbol{P} = \boldsymbol{R}\boldsymbol{Q}. \tag{1.106c}$$

The λ_i, $i = 1, 2, \ldots, n$, are known as the singular values of $\boldsymbol{F}$. Note that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are orthogonal matrices. The singular value decomposition is not unique. For example, one can replace $\boldsymbol{P}$ and $\boldsymbol{Q}$ by $-\boldsymbol{P}$ and $-\boldsymbol{Q}$. The procedure for finding the singular value decomposition of a *nonsingular* matrix is as follows:

1. Find the eigenvalues/eigenvectors $(\lambda_i^2, \boldsymbol{e}_i^*)$, $i = 1, \ldots, n$, of $\boldsymbol{F}^T\boldsymbol{F}$. Construct the square root $\boldsymbol{U} = \sum_{i=1}^{n} \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*$, and its inverse $\boldsymbol{U}^{-1} = \sum_{i=1}^{n} \frac{1}{\lambda_i} \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*$.

2. Find $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$.

3. Construct the factors $\boldsymbol{P}$, $\boldsymbol{\Lambda}$ and $\boldsymbol{Q}$ in the SVD as per Eqns. (1.106).

While finding the singular value decomposition, it is important to construct $\boldsymbol{P}$ as $\boldsymbol{R}\boldsymbol{Q}$ since only then is the constraint $\boldsymbol{f}_i^* = \boldsymbol{R}\boldsymbol{e}_i^*$ met. One should *not* try and construct $\boldsymbol{P}$ *independently* of $\boldsymbol{Q}$ using the eigenvectors of $\boldsymbol{F}\boldsymbol{F}^T$ directly. This will become clear in the example below.

To find the singular decomposition of

$$\boldsymbol{F} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we first find the eigenvalues/eigenvectors $(\lambda_i^2, \boldsymbol{e}_i^*)$ of $\boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{I}$. We get the eigenvalues λ_i as $(1, 1, 1)$, and *choose* the corresponding eigenvectors $\{\boldsymbol{e}_i^*\}$ as $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, where $\{\boldsymbol{e}_i\}$

denotes the canonical basis, so that $\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{e}_1 & | & \boldsymbol{e}_2 & | & \boldsymbol{e}_3 \end{bmatrix} = \boldsymbol{I}$. Since $\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^* = \boldsymbol{I}$, we get $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1} = \boldsymbol{F}$. Lastly, find $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{Q} = \boldsymbol{F}$. Thus the factors in the singular value decomposition are $\boldsymbol{P} = \boldsymbol{F}$, $\boldsymbol{\Lambda} = \boldsymbol{I}$ and $\boldsymbol{Q} = \boldsymbol{I}$. The factors $\boldsymbol{P} = \boldsymbol{I}$, $\boldsymbol{\Lambda} = \boldsymbol{I}$ and $\boldsymbol{Q} = \boldsymbol{F}^T$ are *also* a valid choice corresponding to another choice of $\boldsymbol{Q}$. This again shows that the SVD is nonunique.

Now we show how erroneous results can be obtained if we try to find $\boldsymbol{P}$ independently of $\boldsymbol{Q}$ using the eigenvectors of $\boldsymbol{F}\boldsymbol{F}^T$. Assume that we have already chosen $\boldsymbol{Q} = \boldsymbol{I}$ as outlined above. The eigenvalues of $\boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{I}$ are again $\{1, 1, 1\}$ and if we choose the corresponding eigenvectors $\{\boldsymbol{f}_i^*\}$ as $\{\boldsymbol{e}_i\}$, then we see that we get the wrong result $\boldsymbol{P} = \boldsymbol{I}$, because we have not satisfied the constraint $\boldsymbol{f}_i^* = \boldsymbol{R}\boldsymbol{e}_i^*$.

Now we discuss the SVD for a *singular* matrix, where $\boldsymbol{U}$ is no longer invertible (but still unique), and $\boldsymbol{R}$ is nonunique. First note that from Eqn. (1.104a), we have

$$\boldsymbol{F}\boldsymbol{e}_i^* = \lambda_i \boldsymbol{f}_i^*. \tag{1.107}$$

The procedure for finding the SVD for a singular matrix of dimension n is

1. Find the eigenvalues/eigenvectors $(\lambda_i^2, \boldsymbol{e}_i^*)$, $i = 1, \ldots, n$, of $\boldsymbol{F}^T\boldsymbol{F}$. Let m (where $m < n$) be the number of nonzero eigenvalues λ_i.

2. Find the eigenvectors $\boldsymbol{f}_i^*$, $i = 1, \ldots, m$, corresponding to the nonzero eigenvalues using Eqn. (1.107).

3. Find the eigenvectors $(\boldsymbol{e}_i^*, \boldsymbol{f}_i^*)$, $i = m+1, \ldots, n$, of $\boldsymbol{F}^T\boldsymbol{F}$ and $\boldsymbol{F}\boldsymbol{F}^T$ corresponding to the zero eigenvalues.

4. Construct the factors in the SVD as

$$\boldsymbol{\Lambda} = \mathbf{diag}[\lambda_1, \ldots, \lambda_n], \tag{1.108}$$

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{e}_1^* & | & \boldsymbol{e}_2^* & | & \ldots & | & \boldsymbol{e}_n^* \end{bmatrix}, \tag{1.109}$$

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{f}_1^* & | & \boldsymbol{f}_2^* & | & \ldots & | & \boldsymbol{f}_n^* \end{bmatrix}. \tag{1.110}$$

Obviously, the above procedure will also work if $\boldsymbol{F}$ is nonsingular, in which case $m = n$, and Step (3) in the above procedure is to be skipped.

As an example, consider finding the SVD of

$$\boldsymbol{F} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

The eigenvalues/eigenvectors of $\boldsymbol{F}^T\boldsymbol{F}$ are $(1,1,0)$ and $\boldsymbol{e}_1^* = \boldsymbol{e}_1$, $\boldsymbol{e}_2^* = \boldsymbol{e}_2$ and $\boldsymbol{e}_3^* = \boldsymbol{e}_3$. The eigenvectors $\boldsymbol{f}_i^*$, $i = 1,2$, corresponding to the nonzero eigenvalues are computed using Eqn. (1.107), and are given by

$$\boldsymbol{f}_1^* = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{f}_2^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The eigenvectors $(\boldsymbol{e}_3^*, \boldsymbol{f}_3^*)$ corresponding to the zero eigenvalue of $\boldsymbol{F}^T\boldsymbol{F}$ are $(\boldsymbol{e}_3, \boldsymbol{e}_3)$. Thus, the SVD is given by

$$\boldsymbol{F} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As another example,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}^T.$$

Some of the properties of the SVD are

1. The rank of a matrix (number of linearly independent rows or columns) is equal to the number of non-zero singular values.

2. From Eqn. (1.105), it follows that $\det \boldsymbol{F}$ is nonzero if and only if $\det \boldsymbol{\Lambda} = \Pi_{i=1}^n \lambda_i$ is nonzero. In case $\det \boldsymbol{F}$ is nonzero, then $\boldsymbol{F}^{-1} = \boldsymbol{Q}\boldsymbol{\Lambda}^{-1}\boldsymbol{P}^T$.

3. The condition number λ_1/λ_n (assuming that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$) is a measure of how ill-conditioned the matrix $\boldsymbol{F}$ is. The closer this ratio is to one, the better the conditioning of the matrix is. The larger this value is, the closer $\boldsymbol{F}$ is to being singular. For a singular matrix, the condition number is ∞. For example, the matrix $\mathbf{diag}[10^{-8}, 10^{-8}]$ is *not* ill-conditioned (although its eigenvalues and determinant are small) since its condition number is 1! The SVD can be used to approximate $\boldsymbol{F}^{-1}$ in case $\boldsymbol{F}$ is ill-conditioned.

4. Following a procedure similar to the above, the SVD can be found for a *non-square* matrix $\boldsymbol{F}$, with the corresponding $\boldsymbol{\Lambda}$ also non-square.

1.7 Differentiation of Tensors

The gradient of a scalar field ϕ is defined as

$$\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial x_i}\boldsymbol{e}_i.$$

Similar to the gradient of a scalar field, we define the *gradient of a vector field* $\boldsymbol{v}$ as

$$(\boldsymbol{\nabla}\boldsymbol{v})_{ij} = \frac{\partial v_i}{\partial x_j}.$$

Thus, we have

$$\boldsymbol{\nabla}\boldsymbol{v} = \frac{\partial v_i}{\partial x_j}\boldsymbol{e}_i \otimes \boldsymbol{e}_j.$$

The scalar field

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} := \operatorname{tr}\boldsymbol{\nabla}\boldsymbol{v} = \frac{\partial v_i}{\partial x_j}\operatorname{tr}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = \frac{\partial v_i}{\partial x_j}\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \frac{\partial v_i}{\partial x_j}\delta_{ij} = \frac{\partial v_i}{\partial x_i}, \tag{1.111}$$

is called the divergence of $\boldsymbol{v}$.

The gradient of a second-order tensor $\boldsymbol{T}$ is a third-order tensor defined in a way similar to the gradient of a vector field as

$$\boldsymbol{\nabla}\boldsymbol{T} = \frac{\partial T_{ij}}{\partial x_k}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k.$$

The *divergence of a second-order tensor* $\boldsymbol{T}$, denoted as $\boldsymbol{\nabla}\cdot\boldsymbol{T}$, is defined as

$$\boldsymbol{\nabla}\cdot\boldsymbol{T} = \frac{\partial T_{ij}}{\partial x_j}\boldsymbol{e}_i.$$

The *curl of a vector* $\boldsymbol{v}$, denoted as $\boldsymbol{\nabla}\times\boldsymbol{v}$, is defined by

$$(\boldsymbol{\nabla}\times\boldsymbol{v})\times\boldsymbol{u} := [\boldsymbol{\nabla}\boldsymbol{v} - (\boldsymbol{\nabla}\boldsymbol{v})^T]\boldsymbol{u} \quad \forall \boldsymbol{u} \in V. \tag{1.112}$$

Thus, $\boldsymbol{\nabla}\times\boldsymbol{v}$ is the axial vector corresponding to the skew tensor $[\boldsymbol{\nabla}\boldsymbol{v} - (\boldsymbol{\nabla}\boldsymbol{v})^T]$. In component form, we have

$$\boldsymbol{\nabla}\times\boldsymbol{v} = \epsilon_{ijk}(\boldsymbol{\nabla}\boldsymbol{v})_{kj}\boldsymbol{e}_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}\boldsymbol{e}_i.$$

The *curl of a tensor* $\boldsymbol{T}$, denoted by $\boldsymbol{\nabla}\times\boldsymbol{T}$, is defined by

$$\boldsymbol{\nabla}\times\boldsymbol{T} = \epsilon_{irs}\frac{\partial T_{js}}{\partial x_r}\boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{1.113}$$

The *Laplacian* of a scalar function $\phi(\boldsymbol{x})$ is defined by

$$\boldsymbol{\nabla}^2\phi := \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\phi). \tag{1.114}$$

In component form, the Laplacian is given by

$$\boldsymbol{\nabla}^2\phi = \frac{\partial^2\phi}{\partial x_i \partial x_i}.$$

If $\boldsymbol{\nabla}^2\phi = 0$, then ϕ is said to be harmonic.

The Laplacian of a tensor function $\boldsymbol{T}(\boldsymbol{x})$, denoted by $\boldsymbol{\nabla}^2\boldsymbol{T}$, is defined by

$$(\boldsymbol{\nabla}^2\boldsymbol{T})_{ij} = \frac{\partial^2 T_{ij}}{\partial x_k \partial x_k}.$$

1.7.1 Examples

Although it is possible to derive tensor identities involving differentiation using the above definitions of the operators, the proofs can be quite cumbersome, and hence we prefer to use indicial notation instead. In what follows, $\boldsymbol{u}$ and $\boldsymbol{v}$ are vector fields, and $\boldsymbol{\nabla} \equiv \frac{\partial}{\partial x_i}\boldsymbol{e}_i$ (this is to be interpreted as the 'del' operator acting on a scalar, vector or tensor-valued field, e.g., $\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial x_i}\boldsymbol{e}_i$):

1. Show that
$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \boldsymbol{0}. \tag{1.115}$$

2. Show that
$$\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) = (\boldsymbol{\nabla}\boldsymbol{u})^T\boldsymbol{u}. \tag{1.116}$$

3. Show that $\boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{v}] = (\boldsymbol{\nabla}\boldsymbol{u})^T : \boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{v} \cdot [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u})]$.

4. Show that
$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\boldsymbol{u})^T = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}), \tag{1.117a}$$
$$\boldsymbol{\nabla}^2\boldsymbol{u} := \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\boldsymbol{u}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}). \tag{1.117b}$$

   From Eqns. (1.117a) and (1.117b), it follows that
$$\boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla}\boldsymbol{u}) - (\boldsymbol{\nabla}\boldsymbol{u})^T] = -\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}).$$

   From Eqn. (1.117b), it follows that if $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ and $\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}$, then $\boldsymbol{\nabla}^2\boldsymbol{u} = \boldsymbol{0}$, i.e., $\boldsymbol{u}$ is harmonic.

5. Show that $\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v})$.

6. Let $\boldsymbol{W} \in \text{Skw}$, and let $\boldsymbol{w}$ be its axial vector. Then show that

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{W} &= -\boldsymbol{\nabla} \times \boldsymbol{w}, \\ \boldsymbol{\nabla} \times \boldsymbol{W} &= (\boldsymbol{\nabla} \cdot \boldsymbol{w})\boldsymbol{I} - \boldsymbol{\nabla} \boldsymbol{w}. \end{aligned} \tag{1.118}$$

Solution:

1. Consider the ith component of the left-hand side:

$$\begin{aligned} (\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi)_i &= \epsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \\ &= \epsilon_{ikj} \frac{\partial^2 \phi}{\partial x_k \partial x_j} \qquad \text{(interchanging } j \text{ and } k\text{)} \\ &= -\epsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k}, \end{aligned}$$

which implies that $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \boldsymbol{0}$.

2. $\dfrac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) = \dfrac{1}{2}\dfrac{\partial (u_i u_i)}{\partial x_j}\boldsymbol{e}_j = u_i \dfrac{\partial u_i}{\partial x_j}\boldsymbol{e}_j = (\boldsymbol{\nabla}\boldsymbol{u})^T \boldsymbol{u}$.

3. We have

$$\begin{aligned} \boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{v}] &= \nabla_j ((\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{v})_j \\ &= \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i} v_i\right) \\ &= \frac{\partial v_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + v_i \frac{\partial^2 u_j}{\partial x_i \partial x_j} \\ &= (\boldsymbol{\nabla}\boldsymbol{u})^T : \boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}). \end{aligned}$$

4. The first identity is proved as follows:

$$\begin{aligned} [\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\boldsymbol{u})^T]_i &= \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) \\ &= \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) \\ &= [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u})]_i. \end{aligned}$$

To prove the second identity, consider the last term

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) &= \epsilon_{ijk}\nabla_j(\boldsymbol{\nabla} \times \boldsymbol{u})_k \boldsymbol{e}_i \\
&= \epsilon_{ijk}\nabla_j(\epsilon_{kmn}\nabla_m u_n)\boldsymbol{e}_i \\
&= \epsilon_{ijk}\epsilon_{mnk}\frac{\partial^2 u_n}{\partial x_j \partial x_m}\boldsymbol{e}_i \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\frac{\partial^2 u_n}{\partial x_j \partial x_m}\boldsymbol{e}_i \\
&= \left[\frac{\partial^2 u_j}{\partial x_i \partial x_j} - \frac{\partial^2 u_i}{\partial x_j \partial x_j}\right]\boldsymbol{e}_i \\
&= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \boldsymbol{u}).
\end{aligned}$$

5. We have

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) &= \frac{\partial (\boldsymbol{u} \times \boldsymbol{v})_i}{\partial x_i} \\
&= \epsilon_{ijk}\frac{\partial (u_j v_k)}{\partial x_i} \\
&= \epsilon_{ijk} v_k \frac{\partial u_j}{\partial x_i} + \epsilon_{ijk} u_j \frac{\partial v_k}{\partial x_i} \\
&= \epsilon_{kij} v_k \frac{\partial u_j}{\partial x_i} - \epsilon_{jik} u_j \frac{\partial v_k}{\partial x_i} \\
&= \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}).
\end{aligned}$$

6. Using the relation $W_{ij} = -\epsilon_{ijk} w_k$, we have

$$\begin{aligned}
(\boldsymbol{\nabla} \cdot \boldsymbol{W}) &= -\epsilon_{ijk}\frac{\partial w_k}{\partial x_j}\boldsymbol{e}_i \\
&= -\boldsymbol{\nabla} \times \boldsymbol{w}. \\
(\boldsymbol{\nabla} \times \boldsymbol{W})_{ij} &= \epsilon_{imn}\frac{\partial W_{jn}}{\partial x_m} \\
&= -\epsilon_{imn}\epsilon_{jnr}\frac{\partial w_r}{\partial x_m} \\
&= (\delta_{ij}\delta_{mr} - \delta_{ir}\delta_{mj})\frac{\partial w_r}{\partial x_m} \\
&= \frac{\partial w_r}{\partial x_r}\delta_{ij} - \frac{\partial w_i}{\partial x_j},
\end{aligned}$$

which is the indicial version of Eqn. (1.118).

1.8 The Exponential Function

The exponential of a tensor $\boldsymbol{T}t$ (where $\boldsymbol{T}$ is assumed to be independent of t) can be defined either in terms of its series representation as

$$e^{\boldsymbol{T}t} := \boldsymbol{I} + \boldsymbol{T}t + \frac{1}{2!}(\boldsymbol{T}t)^2 + \cdots , \tag{1.119}$$

or in terms of a solution of the initial value problem

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{T}\boldsymbol{X}(t) = \boldsymbol{X}(t)\boldsymbol{T}, \quad t > 0, \tag{1.120}$$

$$\boldsymbol{X}(0) = \boldsymbol{I}, \tag{1.121}$$

for the tensor function $\boldsymbol{X}(t)$, Note that the superposed dot in the above equation denotes differentiation with respect to t. The existence theorem for linear differential equations tells us that this problem has exactly one solution $\boldsymbol{X} : [0, \infty) \to \text{Lin}$, which we write in the form

$$\boldsymbol{X}(t) = e^{\boldsymbol{T}t}.$$

From Eqn. (1.119), it is immediately evident that

$$e^{\boldsymbol{T}^T t} = (e^{\boldsymbol{T}t})^T, \tag{1.122}$$

and that if $\boldsymbol{A} \in \text{Lin}$ is invertible, then $e^{(\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A})} = \boldsymbol{A}^{-1}\boldsymbol{e}^{\boldsymbol{B}}\boldsymbol{A}$ for all $\boldsymbol{B} \in \text{Lin}$.

Theorem 1.8.1. *For each $t \geq 0$, $e^{\boldsymbol{T}t}$ belongs to Lin$^+$, and*

$$\det(e^{\boldsymbol{T}t}) = e^{(tr\boldsymbol{T})t}. \tag{1.123}$$

Proof. If $(\lambda_i t, \boldsymbol{n}_i)$ is an eigenvalue/eigenvector pair of $\boldsymbol{T}t$, then from Eqn. (1.119), it follows that $(e^{\lambda_i t}, \boldsymbol{n}_i)$ is an eigenvalue/eigenvector pair of $e^{\boldsymbol{T}t}$. Hence, the determinant of $e^{\boldsymbol{T}t}$, which is just the product of the eigenvalues, is given by

$$\det(e^{\boldsymbol{T}t}) = \Pi_{i=1}^n e^{\lambda_i t} = e^{\sum_{i=1}^n \lambda_i t} = e^{(\operatorname{tr}\boldsymbol{T})t}.$$

Since $e^{(\operatorname{tr}\boldsymbol{T})t} > 0$ for all t, $e^{\boldsymbol{T}t} \in \text{Lin}^+$. □

From Eqn. (1.123), it directly follows that

$$\det(e^{\boldsymbol{A}}e^{\boldsymbol{B}}) = \det(e^{\boldsymbol{A}})\det(e^{\boldsymbol{B}}) = e^{\operatorname{tr}\boldsymbol{A}}e^{\operatorname{tr}\boldsymbol{B}} = e^{\operatorname{tr}(\boldsymbol{A}+\boldsymbol{B})} = \det(e^{\boldsymbol{A}+\boldsymbol{B}}).$$

We have

$$\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A} \implies e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{B}}e^{\boldsymbol{A}}. \tag{1.124}$$

However, the converse of the above statement may not be true. Indeed, if $\boldsymbol{AB} \neq \boldsymbol{BA}$, one can have $e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}} = e^{\boldsymbol{B}} = e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{B}}e^{\boldsymbol{A}}$, or $e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{A}+\boldsymbol{B}} \neq e^{\boldsymbol{B}}e^{\boldsymbol{A}}$ or even $e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{B}}e^{\boldsymbol{A}} \neq e^{\boldsymbol{A}+\boldsymbol{B}}$.

As an application of Eqn. (1.124), since $\boldsymbol{T}$ and $-\boldsymbol{T}$ commute, we have $e^{\boldsymbol{T}-\boldsymbol{T}} = \boldsymbol{I} = e^{\boldsymbol{T}}e^{-\boldsymbol{T}}$. Thus,

$$(e^{\boldsymbol{T}})^{-1} = e^{-\boldsymbol{T}}. \tag{1.125}$$

In fact, one can extend this result to get

$$(e^{\boldsymbol{T}})^n = e^{n\boldsymbol{T}} \quad \forall \text{ integer } n.$$

For the exponential of a skew-symmetric tensor, we have the following theorem:

Theorem 1.8.2. *Let $\boldsymbol{W}(t) \in Skw$ for all t. Then $e^{\boldsymbol{W}(t)}$ is a rotation for each $t \geq 0$.*

Proof. By Eqn. (1.125),

$$(e^{\boldsymbol{W}(t)})^{-1} = e^{-\boldsymbol{W}(t)} = e^{\boldsymbol{W}^T(t)} = (e^{\boldsymbol{W}(t)})^T,$$

where the last step follows from Eqn. (1.122). Thus, $e^{\boldsymbol{W}(t)}$ is a orthogonal tensor. By Theorem 1.8.1, $\det(e^{\boldsymbol{W}(t)}) = e^{\operatorname{tr}\boldsymbol{W}(t)} = e^0 = 1$, and hence $e^{\boldsymbol{W}(t)}$ is a rotation. □

In the three-dimensional case, by using the Cayley–Hamilton theorem, we get $\boldsymbol{W}^3(t) = -|\boldsymbol{w}(t)|^2\,\boldsymbol{W}(t)$, where $\boldsymbol{w}(t)$ is the axial vector of $\boldsymbol{W}(t)$. Thus, $\boldsymbol{W}^4(t) = -|\boldsymbol{w}(t)|^2\,\boldsymbol{W}^2(t)$, $\boldsymbol{W}^5(t) = |\boldsymbol{w}(t)|^4\,\boldsymbol{W}(t)$, and so on. Substituting these terms into the series expansion of the exponential function, and using the representations of sine and cosine functions, we get[1]

$$\boldsymbol{R}(t) = e^{\boldsymbol{W}(t)} = \boldsymbol{I} + \frac{\sin(|\boldsymbol{w}(t)|)}{|\boldsymbol{w}(t)|}\boldsymbol{W}(t) + \frac{[1-\cos(|\boldsymbol{w}(t)|)]}{|\boldsymbol{w}(t)|^2}\boldsymbol{W}^2(t). \tag{1.126}$$

Not surprisingly, Eqn. (1.126) has the same form as Eqn. (1.86) with $\alpha = |\boldsymbol{w}(t)|$. Equation (1.126) is known as Rodrigues formula.

[1]Similarly, in the two-dimensional case, if

$$\boldsymbol{W}(t) = \begin{bmatrix} 0 & \gamma(t) \\ -\gamma(t) & 0 \end{bmatrix},$$

where γ is a parameter which is a function of t, then

$$\boldsymbol{R}(t) = e^{\boldsymbol{W}(t)} = \cos\gamma(t)\boldsymbol{I} + \frac{\sin\gamma(t)}{\gamma(t)}\boldsymbol{W}(t) = \begin{bmatrix} \cos\gamma(t) & \sin\gamma(t) \\ -\sin\gamma(t) & \cos\gamma(t) \end{bmatrix}.$$

The exponential tensor $e^{\boldsymbol{T}}$ for a symmetric tensor $\boldsymbol{S}$ is given by

$$e^{\boldsymbol{S}} = \sum_{i=1}^{k} e^{\lambda_i} \boldsymbol{P}_i, \tag{1.127}$$

with $\boldsymbol{P}_i = \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*$ given by

$$\boldsymbol{P}_i(\boldsymbol{S}) = \begin{cases} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{\boldsymbol{S}-\lambda_j \boldsymbol{I}}{\lambda_i-\lambda_j}, & k > 1 \\ \boldsymbol{I}, & k = 1. \end{cases} \tag{1.128}$$

1.9 Divergence, Stokes' and Localization Theorems

We state the divergence, Stokes', potential and localization theorems that are used quite frequently in the following development. The divergence theorem relates a volume integral to a surface integral, while the Stokes' theorem relates a contour integral to a surface integral. Let S represent the surface of a volume V, $\boldsymbol{n}$ represent the unit outward normal to the surface, ϕ a scalar field, $\boldsymbol{u}$ a vector field, and $\boldsymbol{T}$ a second-order tensor field. Then we have

Divergence theorem (also known as the Gauss' theorem)

$$\int_V \boldsymbol{\nabla}\phi \, dV = \int_S \phi \boldsymbol{n} \, dS. \tag{1.129}$$

Applying Eqn. (1.129) to the components u_i of a vector $\boldsymbol{u}$, we get

$$\begin{aligned} \int_V \boldsymbol{\nabla}\cdot\boldsymbol{u} \, dV &= \int_S \boldsymbol{u}\cdot\boldsymbol{n} \, dS, \\ \int_V \boldsymbol{\nabla}\times\boldsymbol{u} \, dV &= \int_S \boldsymbol{n}\times\boldsymbol{u} \, dS, \\ \int_V \boldsymbol{\nabla}\boldsymbol{u} \, dV &= \int_S \boldsymbol{u}\otimes\boldsymbol{n} \, dS. \end{aligned} \tag{1.130}$$

Similarly, on applying Eqn. (1.129) to $\boldsymbol{\nabla}\cdot\boldsymbol{T}$, we get the vector equation

$$\int_V \boldsymbol{\nabla}\cdot\boldsymbol{T} \, dV = \int_S \boldsymbol{T}\boldsymbol{n} \, dS. \tag{1.131}$$

Note that the divergence theorem is applicable even for multiply connected domains provided the surfaces are closed.

Stokes' theorem

Let C be a contour, and S be the area of any arbitrary surface enclosed by the contour C. Then

$$\oint_C \boldsymbol{u} \cdot \boldsymbol{dx} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n} \, dS, \tag{1.132}$$

$$\oint_C \boldsymbol{u} \times \boldsymbol{dx} = \int_S \left[(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{n} - (\boldsymbol{\nabla}\boldsymbol{u})^T \boldsymbol{n} \right] \, dS. \tag{1.133}$$

Localization theorem

If $\int_V \phi \, dV = 0$ for every V, then $\phi = 0$.

Chapter 2

Ordinary differential equations

Consider an n-th order linear ordinary differential equation with constant coefficients. Then by defining new variables, we can convert the n-th order differential equation into a set of n first order ordinary differential equations which can be written in the form

$$\boldsymbol{v}' + \boldsymbol{A}\boldsymbol{v} = \boldsymbol{f}(x), \tag{2.1}$$

where, $\boldsymbol{A}$ is an $n \times n$ (constant) matrix, and $\boldsymbol{v}$ and $\boldsymbol{f}$ are a $n \times 1$ vector. This procedure is illustrated by the following examples.

1. Consider the n-order differential equation

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_{n-1} y' + a_n y = f(x),$$

where the superscript (n) denotes the n'th derivative with respect to x. We can write this equation as

$$\begin{bmatrix} y \\ y' \\ y'' \\ \ldots \\ y^{(n-2)} \\ y^{(n-1)} \end{bmatrix}' + \begin{bmatrix} 0 & -1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & -1 & \ldots & \ldots & 0 \\ 0 & 0 & 0 & -1 & \ldots & 0 \\ & \ldots & & & \ldots & \\ 0 & 0 & \ldots & \ldots & 0 & -1 \\ a_n & a_{n-1} & \ldots & \ldots & a_2 & a_1 \end{bmatrix}_{n\times n} \begin{bmatrix} y \\ y' \\ y'' \\ \ldots \\ y^{(n-2)} \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 0 \\ f(x) \end{bmatrix}.$$

Note that the above equation is in the form given by Eqn. (2.1) with $\boldsymbol{A}$ and $\boldsymbol{f}$ being the square matrix and the vector on the right hand side, respectively.

For the 'sprint-mass-dashpot' governing equation (with t as the independent variable instead of x)

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{f(t)}{m}.$$

we have with $v_1 \equiv x$ and $v_2 \equiv \dot{x}$,

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ \frac{k}{m} & \frac{c}{m} \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} 0 \\ \frac{f(t)}{m} \end{bmatrix}.$$

2. Consider the following system of two second-order differential equations (which can also be written as a single fourth-order equation in either y_1 or y_2 by eliminating y_2 or y_1, respectively):

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2) y_1' + c_2 y_2' - (k_1 + k_2) y_1 + k_2 y_2 + f_1(x), \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + f_2(x). \end{aligned}$$

By defining $v_1 = y_1'$, $v_2 = y_2'$, we have $v_1' = y_1''$ and $v_2' = y_2''$, the above set of equations can be written as

$$\begin{aligned} m_1 v_1' &= -(c_1 + c_2) v_1 + c_2 v_2 - (k_1 + k_2) y_1 + k_2 y_2 + f_1(x), \\ m_2 v_2' &= c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + f_2(x). \end{aligned}$$

Therefore $\{y_1, y_2, v_1, v_2\}$ satisfies the following first order system:

$$\begin{aligned} y_1' &= v_1, \\ y_2' &= v_2, \\ v_1' &= \frac{1}{m_1} \left[-(c_1 + c_2) v_1 + c_2 v_2 - (k_1 + k_2) y_1 + k_2 y_2 \right] + \frac{f_1(x)}{m_1}, \\ v_2' &= \frac{1}{m_2} \left[c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 \right] + \frac{f_2(x)}{m_2}. \end{aligned}$$

From the above examples we see that if we can solve Eqn. (2.1), then we can solve any set of linear ordinary equations with constant coefficients.

Conversely, a system of equations of the form given by Eqn. (2.1) can be converted into n'th order differential equations as follows. Consider the slightly simpler case

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}. \tag{2.2}$$

By differentiating this equation, we get

$$\boldsymbol{y}'' = \boldsymbol{A}\boldsymbol{y}' = \boldsymbol{A}^2 \boldsymbol{y}.$$

By repeatedly differentiating, we get

$$\boldsymbol{y}^{(n)} = \boldsymbol{A}^n \boldsymbol{y}, \tag{2.3}$$

where the superscript (n) denotes the n'th derivative. It follows that

$$\boldsymbol{y}^{(n)} - I_1\boldsymbol{y}^{(n-1)} + \ldots + (-1)^n I_n\boldsymbol{y} = (\boldsymbol{A}^n - I_1\boldsymbol{A}^{n-1} + \ldots + (-1)^n I_n\boldsymbol{I})\boldsymbol{y} = \boldsymbol{0}, \tag{2.4}$$

where the last step follows from the Cayley-Hamilton theorem. Thus, we see that $\{y_1, y_2, \ldots y_n\}$ all obey the same differential equation, and hence have the same solution form but with *different* constants. Let the initial conditions for Eqn. (2.2) be given as $\boldsymbol{y}(0) = \boldsymbol{y}_0$. Then the n initial conditions required for solving Eqn. (2.4) are obtained using Eqn. (2.3) as $\boldsymbol{y}(0) = \boldsymbol{y}_0$, $\boldsymbol{y}'(0) = \boldsymbol{A}\boldsymbol{y}_0$, $\ldots$, $\boldsymbol{y}^{(n-1)}(0) = \boldsymbol{A}^{n-1}\boldsymbol{y}_0$.

2.1 General solution for Eqn. (2.1)

Multiplying Eqn. (2.1) by $e^{\boldsymbol{A}x}$, we get

$$e^{\boldsymbol{A}x}\boldsymbol{v}' + e^{\boldsymbol{A}x}\boldsymbol{A}\boldsymbol{v} = e^{\boldsymbol{A}x}\boldsymbol{f}(x).$$

or

$$\frac{d}{dx}\left(e^{\boldsymbol{A}x}\boldsymbol{v}\right) = e^{\boldsymbol{A}x}\boldsymbol{f}(x).$$

Let $x \in [a, b]$. Integrating the above equation, we get

$$e^{\boldsymbol{A}x}\boldsymbol{v} = \int_a^x e^{\boldsymbol{A}\xi}\boldsymbol{f}(\xi)\,d\xi + \boldsymbol{c},$$

where $\boldsymbol{c}$ is a constant vector independent of x. Multiplying the above equation by $[e^{\boldsymbol{A}x}]^{-1} = e^{-\boldsymbol{A}x}$, we get

$$\boldsymbol{v}(x) = \int_a^x e^{-\boldsymbol{A}(x-\xi)}\boldsymbol{f}(\xi)\,d\xi + e^{-\boldsymbol{A}x}\boldsymbol{c}, \tag{2.5a}$$

$$= \int_0^{x-a} e^{-\boldsymbol{A}\xi}\boldsymbol{f}(x-\xi)\,d\xi + e^{-\boldsymbol{A}x}\boldsymbol{c}, \tag{2.5b}$$

where the second equation is obtained from the first one by a change of variable.

The constant vector $\boldsymbol{c}$ is found using the boundary conditions in a *boundary value problem* or using the initial conditions in an *initial value problem* (An *initial boundary value problem* would involve partial differential equations and is out of scope of the current chapter). The independent variable is usually denoted by x in the former case, and by t (for time) in the latter, where $t \in [0, T]$, so that Eqns. (2.5) would be typically written as

$$\boldsymbol{v}(t) = \int_0^t e^{-\boldsymbol{A}(t-\xi)}\boldsymbol{f}(\xi)\,d\xi + e^{-\boldsymbol{A}t}\boldsymbol{c},$$

$$= \int_0^t e^{-\boldsymbol{A}\xi} \boldsymbol{f}(t-\xi)\, d\xi + e^{-\boldsymbol{A}t} \boldsymbol{c}. \tag{2.6}$$

If $\boldsymbol{v}(0) = \boldsymbol{0}$, then from the above equation, we see that $\boldsymbol{c} = 0$, and the solution is given by

$$\begin{aligned} \boldsymbol{v}(t) &= \int_0^t e^{-\boldsymbol{A}(t-\xi)} \boldsymbol{f}(\xi)\, d\xi, \\ &= \int_0^t e^{-\boldsymbol{A}\xi} \boldsymbol{f}(t-\xi)\, d\xi. \end{aligned} \tag{2.7}$$

As an example, if one considers the beam bending equation

$$EI \frac{d^2 w}{dx^2} = M_b,$$

then the constant vector $\boldsymbol{c} = (c_1, c_2)$ (there are two constants since the order of the differential equation is two), is found using the boundary conditions at the ends of the beam. For example, if the beam is simply-supported, then the boundary conditions will be $w(0) = w(L) = 0$.

On the other hand, if we consider the equation for a spring-mass-damper

$$m\ddot{x} + c\dot{x} + kx = f(t),$$

then the vector $\boldsymbol{c} = (c_1, c_2)$ would be found using the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$.

One case where the exponential matrix can be computed rather easily is if $\boldsymbol{A} \in \text{Sym}$. Then, assuming $\boldsymbol{A}$ to be a constant matrix, and using the spectral decomposition

$$\boldsymbol{A} = \sum_{i=1}^{n} \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*,$$

we get

$$e^{-\boldsymbol{A}\xi} = \sum_{i=1}^{n} e^{-\lambda_i \xi} \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*.$$

Substituting into Eqn. (2.5b), we get

$$\boldsymbol{v}(x) = \sum_{i=1}^{n} \boldsymbol{e}_i^* \left\{ \int_0^{x-a} e^{-\lambda_i \xi} \left[\boldsymbol{f}(x-\xi) \cdot \boldsymbol{e}_i^* \right] d\xi + e^{-\lambda_i x} (\boldsymbol{c} \cdot \boldsymbol{e}_i^*) \right\}. \tag{2.8}$$

Similarly, if $\boldsymbol{A}$ possesses n linearly independent eigenvectors (if all the eigenvalues are distinct then the eigenvectors are linearly independent, although the converse may not be true as is evident from the case of a symmetric tensor with repeated eigenvalues), then we can write

$$\boldsymbol{A} = \sum_{i=1}^{n} \lambda_i \boldsymbol{u}_i \otimes \boldsymbol{v}_i,$$

where $(\boldsymbol{u}_i, \boldsymbol{v}_i)$, $i = 1, 2, \ldots, n$, are the eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{A}^T$, i.e.,

$$\begin{aligned} \boldsymbol{A}\boldsymbol{u}_i &= \lambda_i \boldsymbol{u}_i, \\ \boldsymbol{A}^T \boldsymbol{v}_i &= \lambda_i \boldsymbol{v}_i, \end{aligned}$$

and $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are normalized such that $\boldsymbol{u}_i \cdot \boldsymbol{v}_j = \delta_{ij}$. From a practical point of view

$$\begin{bmatrix} \boldsymbol{v}_1 \mid \boldsymbol{v}_2 \mid \ldots \mid \boldsymbol{v}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 \\ -- \\ \boldsymbol{u}_2 \\ -- \\ -- \\ \boldsymbol{u}_n \end{bmatrix}^{-1},$$

with $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, etc. placed along columns, and $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ etc. placed along the rows in the left and right hand side matrices, so that one can compute the eigenvectors of $\boldsymbol{A}^T$ simply by the above inversion. Thus,

$$e^{-\boldsymbol{A}\xi} = \sum_{i=1}^{n} e^{-\lambda_i \xi} \boldsymbol{u}_i \otimes \boldsymbol{v}_i.$$

Substituting into Eqn. (2.5b), we get (do not confuse between $\boldsymbol{v}$ and $\boldsymbol{v}_i$)

$$\begin{aligned} \boldsymbol{v}(x) &= \sum_{i=1}^{n} \boldsymbol{u}_i \left\{ \int_0^{x-a} e^{-\lambda_i \xi} \left[\boldsymbol{f}(x-\xi) \cdot \boldsymbol{v}_i \right] d\xi + e^{-\lambda_i x} (\boldsymbol{c} \cdot \boldsymbol{v}_i) \right\} \\ &= \sum_{i=1}^{n} \boldsymbol{u}_i \left\{ \int_0^{x-a} e^{-\lambda_i \xi} \left[\boldsymbol{f}(x-\xi) \cdot \boldsymbol{v}_i \right] d\xi + e^{-\lambda_i x} c_i \right\}, \end{aligned} \tag{2.9}$$

where $c_i := \boldsymbol{c} \cdot \boldsymbol{v}_i$ are constants (If two eigenvectors $\boldsymbol{v}_i$ are complex conjugates, then the corresponding constants c_i are also complex conjugates), and the constant a in the integration limit can be taken to be zero if the integrals are well-behaved.

However, instead of finding the solution using Eqn. (2.9), a better way is as follows. From Eqn. (2.9), we see that the solution to the *homogeneous* equation is

$$\boldsymbol{v}(x) = \sum_{i=1}^{n} c_i e^{-\lambda_i x} \boldsymbol{u}_i,$$

which can be written in matrix form as

$$\boldsymbol{v}(x) = \boldsymbol{X}\boldsymbol{c},$$

where

$$\boldsymbol{X} = \left[e^{-\lambda_1 x}\boldsymbol{u}_1 \mid e^{-\lambda_2 x}\boldsymbol{u}_2 \mid \ldots \mid e^{-\lambda_n x}\boldsymbol{u}_n \right], \quad \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ \ldots \\ c_n \end{bmatrix}.$$

To solve the inhomogeneous system, we use the method of variation of parameters, where we assume the solution to be $\boldsymbol{v}(x) = \boldsymbol{X}(x)\boldsymbol{z}(x)$. Substituting this solution form into Eqn. (2.1) and using the fact that $(\boldsymbol{X}' + \boldsymbol{A}\boldsymbol{X})\boldsymbol{z} = \boldsymbol{0}$, we get

$$\boldsymbol{X}\boldsymbol{z}' = \boldsymbol{f}.$$

Since the columns of $\boldsymbol{X}$ are linearly independent, it is invertible, so that

$$\boldsymbol{z} = \int_a^x \boldsymbol{X}^{-1}(\xi)\boldsymbol{f}(\xi)\, d\xi + \boldsymbol{c}.$$

Thus, the complete solution is

$$\boldsymbol{v}(x) = \boldsymbol{X}\int_a^x \boldsymbol{X}^{-1}(\xi)\boldsymbol{f}(\xi)\, d\xi + \boldsymbol{X}\boldsymbol{c}. \tag{2.10}$$

As an example, consider the solution of the following set of equations

$$\begin{aligned} \frac{dx}{dt} &= 3x + z + e^{2t}, \\ \frac{dy}{dt} &= -x + 4y + z + e^{2t}, \\ \frac{dz}{dt} &= 4x - 4y + 2z - e^{2t}. \end{aligned}$$

In this case, we have

$$\boldsymbol{A} = \begin{bmatrix} -3 & 0 & -1 \\ 1 & -4 & -1 \\ -4 & 4 & -2 \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} e^{2t} \\ e^{2t} \\ -e^{2t} \end{bmatrix},$$

$$
\begin{aligned}
\boldsymbol{u}_1 &= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \\
\lambda_i &= \{-4, -3, -2\}. \\
\boldsymbol{X} &= \begin{bmatrix} 2e^{4t} & e^{3t} & -e^{2t} \\ e^{4t} & e^{3t} & -e^{2t} \\ 2e^{4t} & 0 & e^{2t} \end{bmatrix}, \quad \boldsymbol{X}^{-1} = \begin{bmatrix} e^{-4t} & -e^{-4t} & 0 \\ -3e^{-3t} & 4e^{-3t} & e^{-3t} \\ -2e^{-2t} & 2e^{-2t} & e^{-2t} \end{bmatrix}.
\end{aligned}
$$

Substituting into the solution given by Eqn. (2.10), we get

$$
\begin{aligned}
x &= e^{2t}t + 2c_1e^{4t} + c_2e^{3t} - c_3e^{2t}, \\
y &= e^{2t}t + c_1e^{4t} + c_2e^{3t} - c_3e^{2t}, \\
z &= -e^{2t}t + 2c_1e^{4t} + c_3e^{2t}.
\end{aligned}
$$

As another example, if

$$
\begin{aligned}
\boldsymbol{A} &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
\boldsymbol{u}_1 &= \begin{bmatrix} \frac{1-i\sqrt{3}}{2} \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} \frac{1+i\sqrt{3}}{2} \\ 1 \end{bmatrix}, \\
\lambda_i &= \left\{ \frac{3+i\sqrt{3}}{2}, \frac{3-i\sqrt{3}}{2} \right\}, \\
\boldsymbol{X} &= \begin{bmatrix} \frac{(1-i\sqrt{3})e^{\frac{-(3+i\sqrt{3})t}{2}}}{2} & \frac{(1+i\sqrt{3})e^{\frac{-(3-i\sqrt{3})t}{2}}}{2} \\ e^{\frac{-(3+i\sqrt{3})t}{2}} & e^{\frac{-(3-i\sqrt{3})t}{2}} \end{bmatrix}, \quad \boldsymbol{X}^{-1} = \begin{bmatrix} \frac{i}{\sqrt{3}}e^{\frac{(3+i\sqrt{3})t}{2}} & \frac{3-i\sqrt{3}}{6}e^{\frac{(3+i\sqrt{3})t}{2}} \\ -\frac{i}{\sqrt{3}}e^{\frac{(3-i\sqrt{3})t}{2}} & \frac{3+i\sqrt{3}}{6}e^{\frac{(3-i\sqrt{3})t}{2}} \end{bmatrix}.
\end{aligned}
$$

Substituting into the solution given by Eqn. (2.10), we get

$$
\begin{aligned}
v_1 &= \frac{1}{3} + \frac{(1-i\sqrt{3})(3c_1-1) + (1+i\sqrt{3})(3c_2-1)e^{i\sqrt{3}t}}{6e^{(3+i\sqrt{3})t/2}}, \\
v_2 &= \frac{2}{3} + \frac{3c_1 - 1 + (3c_2-1)e^{i\sqrt{3}t}}{3e^{(3+i\sqrt{3})t/2}},
\end{aligned}
$$

where c_1 and c_2 are complex conjugates.

2.2 Generalization of Eqn. (2.1)

In Eqn. (2.1), the matrix $\boldsymbol{A}$ was assumed to be a constant, i.e., independent of x. Now we consider a generalization where $\boldsymbol{A}$ is a function of x, so that Eqn. (2.1) becomes

$$\boldsymbol{v}' + \boldsymbol{A}(x)\boldsymbol{v} = \boldsymbol{f}(x). \tag{2.11}$$

Note that the above set of equations is linear but with variable coefficients. In order to obtain a closed-form solution, we assume that $\boldsymbol{A}(x)$ satisfies the constraint

$$\boldsymbol{A}(x)\int_a^x \boldsymbol{A}(\eta)\,d\eta = \left(\int_a^x \boldsymbol{A}(\eta)\,d\eta\right)\boldsymbol{A}(x). \tag{2.12}$$

One can show that the above condition is satisfied if and only if

$$\frac{d(e^{\int_a^x \boldsymbol{A}(\eta)\,d\eta})}{dx} = \boldsymbol{A}(x)e^{\int_a^x \boldsymbol{A}(\eta)\,d\eta} = e^{\int_a^x \boldsymbol{A}(\eta)\,d\eta}\boldsymbol{A}(x). \tag{2.13}$$

Our set of differential equations in place of Eqns. (2.1) is

$$\boldsymbol{v}' + \boldsymbol{A}(x)\boldsymbol{v} = \boldsymbol{f}(x), \tag{2.14}$$

Multiplying by $e^{\int_a^x \boldsymbol{A}(\eta)\,d\eta}$, we get

$$e^{\int_a^x \boldsymbol{A}(\eta)\,d\eta}\boldsymbol{v}' + e^{\int_a^x \boldsymbol{A}(\eta)\,d\eta}\boldsymbol{A}(x)\boldsymbol{v} = e^{\int_a^x \boldsymbol{A}(\eta)\,d\eta}\boldsymbol{f}(x).$$

or,

$$\frac{d}{dx}\left(e^{\int_a^x \boldsymbol{A}(\eta)\,d\eta}\boldsymbol{v}\right) = e^{\int_a^x \boldsymbol{A}(\eta)\,d\eta}\boldsymbol{f}(x).$$

Integrating the above equation, we get

$$e^{\int_a^x \boldsymbol{A}(\eta)\,d\eta}\boldsymbol{v} = \int_a^x e^{\int_a^\xi \boldsymbol{A}(\eta)\,d\eta}\boldsymbol{f}(\xi)\,d\xi + \boldsymbol{c},$$

where $\boldsymbol{c}$ is a constant vector, again to be determined from the boundary or initial conditions. Thus, finally we have,

$$\boldsymbol{v} = e^{-\int_a^x \boldsymbol{A}(\eta)\,d\eta}\left[\int_a^x e^{\int_a^\xi \boldsymbol{A}(\eta)\,d\eta}\boldsymbol{f}(\xi)\,d\xi + \boldsymbol{c}\right]. \tag{2.15}$$

If the matrices $\int_a^x \boldsymbol{A}(\eta)\,d\eta$ and $\int_a^\xi \boldsymbol{A}(\eta)\,d\eta$ commute[1] then the above equation simplifies to

$$\boldsymbol{v}(x) = \int_a^x e^{-\int_\xi^x \boldsymbol{A}(\eta)\,d\eta} \boldsymbol{f}(\xi)\,d\xi + e^{-\int_a^x \boldsymbol{A}(\eta)\,d\eta} \boldsymbol{c} \tag{2.16a}$$

$$= \int_0^{x-a} e^{-\int_{x-\xi}^x \boldsymbol{A}(\eta)\,d\eta} \boldsymbol{f}(x-\xi)\,d\xi + e^{-\int_a^x \boldsymbol{A}(\eta)\,d\eta} \boldsymbol{c}. \tag{2.16b}$$

Note that Eqns. (2.16) reduce to Eqns. (2.5) (by redefining the constant $\boldsymbol{c}$) when $\boldsymbol{A}(\eta)$ is a constant matrix. As usual, in initial value problems, we prefer to use the notation $t \in [0, T]$ for the independent variable, and then Eqns. (2.16) become

$$\boldsymbol{v}(x) = \int_0^t e^{-\int_\xi^t \boldsymbol{A}(\eta)\,d\eta} \boldsymbol{f}(\xi)\,d\xi + e^{-\int_a^t \boldsymbol{A}(\eta)\,d\eta} \boldsymbol{c} \tag{2.17a}$$

$$= \int_0^{t-a} e^{-\int_{t-\xi}^t \boldsymbol{A}(\eta)\,d\eta} \boldsymbol{f}(t-\xi)\,d\xi + e^{-\int_a^t \boldsymbol{A}(\eta)\,d\eta} \boldsymbol{c}. \tag{2.17b}$$

The main difficulty with the above solution procedure is that in most cases, the matrix $\boldsymbol{A}$ will not satisfy the constraint given by Eqn. (2.12) (except in the case where $\boldsymbol{A}(x)$ is a 1×1 matrix as in the following section, in which case the constraint is always met). Due to this difficulty, we try and solve second and higher-order differential equations (especially ones with variable coefficients) independently.

2.3 Linear first order ordinary differential equations

For a first order differential equation the matrix $\boldsymbol{A}(x)$ in Eqn. (2.11) is simply a 1×1 matrix, i.e., $\boldsymbol{A}(x) = [A(x)]$ so that the constraint given by Eqn. (2.12) is automatically satisfied, and hence, the derived solutions are also valid. The differential equation given by Eqn. (2.11) reduces to

$$v'(x) + A(x)v = f(x),$$

while the solutions given by Eqns. (2.16) reduce to

$$v(x) = \int_a^x e^{-\int_\xi^x A(\eta)\,d\eta} f(\xi)\,d\xi + ce^{-\int_a^x A(\eta)\,d\eta} \tag{2.18a}$$

[1]Note that Eqn. (2.12) *does not* imply that $\int_a^x \boldsymbol{A}(\eta)\,d\eta$ and $\int_a^\xi \boldsymbol{A}(\eta)\,d\eta$ commute as the following counterexample (with $a = 0$) shows:

$$\boldsymbol{A}(\eta) = \begin{bmatrix} 4\eta^3 & 5\eta^4 & 6\eta^5 \\ -6\eta^2 & -8\eta^3 & -10\eta^4 \\ 2\eta & 3\eta^2 & 4\eta^3 \end{bmatrix}.$$

$$= \int_0^{x-a} e^{-\int_{x-\xi}^x A(\eta)\,d\eta} f(x-\xi)\,d\xi + ce^{-\int_a^x A(\eta)\,d\eta}, \tag{2.18b}$$

where c is a constant to be determined. In initial value problems, we prefer to use t as the independent variable, and in place of Eqns. (2.18), we have

$$v(t) = \int_0^t e^{-\int_\xi^t A(\eta)\,d\eta} f(\xi)\,d\xi + ce^{-\int_a^t A(\eta)\,d\eta} \tag{2.19a}$$

$$= \int_0^t e^{-\int_{t-\xi}^t A(\eta)\,d\eta} f(t-\xi)\,d\xi + ce^{-\int_a^t A(\eta)\,d\eta}. \tag{2.19b}$$

We now illustrate the application of Eqn. (2.18) (or Eqn. (2.19)) to various examples.

1. In the differential equation
$$y' + 2y = x^3 e^{-2x},$$
we see that $A(x) = 2$ and $f(x) = x^3 e^{-2x}$. Thus, from Eqn. (2.18a), we get
$$\begin{aligned} y(x) &= \int_a^x e^{-2(x-\xi)} \xi^3 e^{-2\xi}\,d\xi + ce^{-2(x-a)} \\ &= \frac{e^{-2x}}{4}\left[x^4 + c_0\right], \end{aligned}$$
where c_0 is to be determined from the initial condition.

2. In the differential equation
$$y' + (\cot x)y = x \csc x,$$
we have $A(x) = \cot x$, and $f(x) = x \csc x$. Thus, from Eqn. (2.18a), we have
$$\begin{aligned} y(x) &= \int_a^x e^{-\int_\xi^x \cot\eta\,d\eta} \xi \csc\xi\,d\xi + ce^{-\int_a^x \cot\eta\,d\eta} \\ &= \frac{\csc x}{2}\left[x^2 + c_0\right], \end{aligned}$$
where c_0 is a constant.

3. In the differential equation
$$y' - 2xy = 1,$$
we have $A(x) = -2x$, and $f(x) = 1$. Thus, from Eqn. (2.18a), we have
$$\begin{aligned} y(x) &= \int_a^x e^{\int_\xi^x 2\eta\,d\eta}\,d\xi + ce^{\int_a^x 2\eta\,d\eta} \\ &= e^{x^2}\left[\int_0^x e^{-\xi^2}\,d\xi + c_0\right]. \end{aligned}$$
If the initial condition is $y(0) = y_0$, we get $c_0 = y_0$.

2.4 Nonlinear first order ordinary differential equations

We now discuss the solution of nonlinear first order differential equations.

2.4.1 Separable equations

A first order differential equation is separable if it can be written as

$$h(y)y' = g(x).$$

The method of solution is best illustrated by examples.

1. Solve

$$y' = x(1 + y^2).$$

   Write this as

$$\frac{dy}{1 + y^2} = x\,dx.$$

   Integrating, we get

$$\tan^{-1} y = \frac{x^2}{2} + c.$$

2. Solve

$$y' = -\frac{x}{y}, \quad y(1) = 1.$$

   Write this as

$$y\,dy = -x\,dx,$$

   Integrating, we get

$$x^2 + y^2 = c^2,$$

   where we have taken the constant of integration to be positive since the LHS is positive. Thus,

$$y = \pm\sqrt{c^2 - x^2}. \tag{2.20}$$

   Since $y(1) = 1$, we get $c^2 = 2$, or,

$$y = \sqrt{2 - x^2}, \quad -\sqrt{2} \leq x \leq \sqrt{2}.$$

   If the initial condition were $y(1) = -1$, then we take the other root in Eqn. (2.20), and the solution is

$$y = -\sqrt{2 - x^2} \quad -\sqrt{2} \leq x \leq \sqrt{2}.$$

3. Solve
$$y' = \frac{2x+1}{5y^4+1}.$$
Write this as
$$(5y^4+1)dy = (2x+1)dx.$$
Integrating, we get
$$y^5 + y = x^2 + x + c.$$
Note that it is not possible to explicitly solve for y as a function of x in this case.

4. Solve
$$y' = 2xy^2, \quad y(0) = y_0. \tag{2.21}$$
Write this as
$$\frac{dy}{y^2} = 2x\,dx.$$
Note that while writing the above, we are implicitly assuming that $y \neq 0$. Integrating, we get
$$-\frac{1}{y} = x^2 + c,$$
or
$$y = -\frac{1}{x^2+c}. \tag{2.22}$$
Thus, $y = 0$ and the above solution are *both* solutions of Eqn. (2.21).

Imposing the initial condition in Eqn. (2.22), we get
$$y = \frac{y_0}{1 - y_0 x^2}.$$
If $y_0 < 0$, then the above solution is valid on $x \in (-\infty, \infty)$. If $y_0 = 0$, then $y = 0$ is the solution, while if $y_0 > 0$, then the above solution is valid only for $x \in (-1/\sqrt{y_0}, 1/\sqrt{y_0})$. This example shows that the range of validity of the solution can depend on the initial condition.

5. Find all solutions of
$$y' = \frac{x}{2}(1-y^2).$$
Write this as
$$\frac{2dy}{1-y^2} = x\,dx.$$
Implicitly, we are assuming that $y \neq \pm 1$. Writing the above equation as
$$\left[\frac{1}{y-1} - \frac{1}{y+1}\right] dy = -x\,dx,$$

and integrating, we get

$$\log \frac{y-1}{y+1} = -\frac{x^2}{2} + k,$$

or, alternatively,

$$\frac{y-1}{y+1} = ce^{-x^2/2}.$$

Thus, $y = 1$, $y = -1$ and the above solution are all solutions of the differential equation.

2.4.2 Exact nonlinear first order equations

Write the differential equation in the form

$$M(x, y)\, dx + N(x, y)\, dy = 0. \tag{2.23}$$

Note that $F(x, y) = 0$ is an implicit solution of the differential equation

$$F_x(x, y)\, dx + F_y(x, y)\, dy = 0. \tag{2.24}$$

Eqn. (2.23) is said to be exact if there exists a function $F(x, y)$ such that

$$F_x(x, y) = M(x, y), \tag{2.25a}$$

$$F_y(x, y) = N(x, y). \tag{2.25b}$$

Since $F_{xy} = F_{yx}$, a necessary condition for an equation to be exact is that (it can be proved that this condition is sufficient also)

$$M_y = N_x. \tag{2.26}$$

Thus, a procedure for solving an exact equation is as follows:

1. Check that $M_y = N_x$. If not, then the following procedure cannot be applied.
2. Integrate Eqn. (2.25a) with respect to x to get

$$F(x, y) = G(x, y) + \phi(y). \tag{2.27}$$

3. Differentiate Eqn. (2.27) with respect to y, and combine with Eqn. (2.25b) to get

$$\phi'(y) = N(x, y) - G_y(x, y) \tag{2.28}$$

4. Integrate the above equation with respect to y, taking the constant of integration to be zero and substitute into Eqn. (2.27) to obtain $F(x, y)$.

Alternatively, using the Leibnitz rule and Eqn. (2.26), one can verify that the solution for $F(x, y)$ that satisfies Eqns. (2.25) is given by

$$F(x,y) = \int_{x_0}^{x} M(\xi, y_0)\, d\xi + \int_{y_0}^{y} N(x,\eta)\, d\eta \tag{2.29a}$$

$$= \int_{x_0}^{x} M(\xi, y)\, d\xi + \int_{y_0}^{y} N(x_0,\eta)\, d\eta. \tag{2.29b}$$

Let us consider a few examples.

1. Solve

$$e^{xy}\left[y\tan x + \sec^2 x\right]\, dx + xe^{xy}\tan x\, dy = 0. \tag{2.30}$$

Note that one should not 'cancel' the e^{xy} factor, since without this factor, the equation is not exact! Verify that $M_y = N_x$. Thus,

$$F_x(x,y) = e^{xy}\left[y\tan x + \sec^2 x\right], \tag{2.31a}$$

$$F_y(x,y) = xe^{xy}\tan x. \tag{2.31b}$$

It is easier to integrate Eqn. (2.31b). We get

$$F(x,y) = e^{xy}\tan x + \phi(x).$$

Differentiating this with respect to x yields

$$F_x(x,y) = e^{xy}\left[y\tan x + \sec^2 x\right] + \phi'(x).$$

Comparing against Eqn. (2.31a), we get $\phi'(x) = 0$ or ϕ is a constant which can be taken to be zero. Thus, using Eqn. (2.29a), we get

$$e^{xy}\tan x = c, \tag{2.32}$$

or by redefining the constant c,

$$y = \frac{1}{x}\left[\log\cot x + c\right]. \tag{2.33}$$

Alternatively, by directly using either Eqn. (2.29a) or (2.29b), we get $F = e^{xy}\tan x - e^{x_0 y_0}\tan x_0 = 0$, so that

$$e^{xy}\tan x = e^{x_0 y_0}\tan x_0,$$

which is of the same form as Eqn. (2.32).

If one cancels e^{xy} in Eqn. (2.30), then we get

$$\frac{dy}{dx} + \frac{y}{x} = -\frac{\sec^2 x}{x\tan x},$$

which is a linear first order differential equation whose solution is given by Eqn. (2.18a). One can verify that one gets the same solution as Eqn. (2.33).

2. Solve
$$\left[e^{xy}x^3(xy+4)+3y\right]\,dx+\left[x^5e^{xy}+3x\right]\,dy=0.$$
Since $M=e^{xy}x^3(xy+4)+3y$ and $N=x^5e^{xy}+3x$, we see that $M_y=N_x$, and the equation is exact. From either of Eqns. (2.29), we get
$$F=x^4e^{xy}+3xy-x_0^4e^{x_0y_0}-3x_0y_0=0,$$
or, alternatively,
$$x^4e^{xy}+3xy=c.$$

2.4.3 Making equations exact by using integrating factors

Sometimes, even if an equation is not exact, it can be made exact by multiplying the governing differential equation by an integrating factor, i.e.,
$$\mu(x,y)M(x,y)\,dx+\mu(x,y)N(x,y)\,dy=0. \tag{2.34}$$
By replacing M and N by μM and μN in Eqn. (2.26), we get the governing equation for the integrating factor μ as
$$\frac{\partial(\mu M)}{\partial y}=\frac{\partial(\mu N)}{\partial x},$$
or, equivalently,
$$N\mu_x-M\mu_y=\mu(M_y-N_x). \tag{2.35}$$
Since μ appears on both sides of the above equation, it is more convenient to write it as $e^{g(x,y)}$, so that we get
$$Ng_x-Mg_y=M_y-N_x. \tag{2.36}$$
Solving the above equation yields the integrating factor to make the governing differential equation exact, which can then be solved as shown in Section 2.4.2.

The procedure for solving Eqn. (2.36) is as follows:

1. Check if $(M_y-N_x)/N$ is a function of x alone. If it is, then from Eqn. (2.36), we see that $g=g(x)$ can be obtained by solving the ordinary differential equation
$$\frac{dg}{dx}=\frac{M_y-N_x}{N}.$$
.

2. Check if $(M_y-N_x)/M$ is a function of y alone. If it is, then from Eqn. (2.36), we see that $g=g(y)$ can be obtained by solving the ordinary differential equation
$$\frac{dg}{dy}=-\frac{M_y-N_x}{M}.$$
.

3. If both the above checks fail, then $g = g(x, y)$ is a function of both x and y. By substituting for M and N in a given problem, and matching the coefficients of some common terms on both sides of Eqn. (2.36), we obtain equations for g_x and g_y which are then integrated to find g; this procedure generally works better than assuming g to be either $P(x)Q(y)$ or $P(x)+Q(y)$, etc. In some cases such as in the Riccati equation presented below, it may not be possible to find the integrating factor analytically except for some special choices of coefficients.

We now present several examples:

1. An equation is said to be homogeneous (not to be confused with equations of the type $y' + A(x)y = 0$ which are also termed homogeneous) if it is of the type

$$q(y/x)\, dx - dy = 0. \tag{2.37}$$

Thus, $M = q(y/x)$ and $N = -1$. Let $u = y/x$, i.e., $y = ux$, so that $dy = xdu + udx$. Substituting into Eqn. (2.37), we get

$$[q(u) - u]\; dx - xdu = 0,$$

which can be written in the form

$$\frac{du}{q(u) - u} = \frac{dx}{x}.$$

Thus,

$$\log x = \int \frac{du}{q(u) - u} + c.$$

As an example, an equation of the type

$$(ax + by + c)dx + (\bar{a}x + \bar{b}y + \bar{c})dy = 0, \tag{2.38}$$

is not homogeneous, but can be made homogeneous by a transformation of the type $x = X + \xi$ and $y = Y + \eta$, where (ξ, η) are determined so that the equation becomes of the type

$$(aX + bY)dX + (\bar{a}X + \bar{b}Y)dY = 0. \tag{2.39}$$

Thus, we find (ξ, η) that are solutions of the equations

$$a\xi + b\eta = -c, \tag{2.40a}$$

$$\bar{a}\xi + \bar{b}\eta = -\bar{c}. \tag{2.40b}$$

Eqn. (2.39) is a homogeneous equation since it can be written as

$$\left(a + b\frac{Y}{X}\right) dX + \left(\bar{a} + \bar{b}\frac{Y}{X}\right) dY = 0.$$

If $\bar{a} = \alpha a$ and $\bar{b} = \alpha b$, then Eqns. (2.40) cannot be solved for ξ and η. In this case, we write Eqn. (2.38) as

$$(ax + by + c)dx + (\alpha(ax + by) + \bar{c})dy = 0, \tag{2.41}$$

Let $w = ax + by$, so that $dy = (dw - adx)/b$. Substituting into the above equation, we have

$$[b(w + c) - a]\ dx + (\alpha w + \bar{c})\ dw = 0,$$

which can be solved since it can be written as

$$\int \frac{(\alpha w + \bar{c})dw}{[b(w + c) - a]} = -x + c.$$

2. A Bernoulli equation is an equation of the form

$$y' + A(x)y = f(x)y^r, \tag{2.42}$$

where $r \neq 1$. We see that $M = A(x)y - f(x)y^r$, $N = 1$. Thus, carrying out the first two checks in the above procedure, we see that $g = g(x, y)$. From Eqn. (2.36), we see that

$$g_x - [A(x)y - f(x)y^r]\, g_y = A(x) - rf(x)y^{r-1}. \tag{2.43}$$

By matching the coefficient of $f(x)$, we get

$$y^r g_y = -ry^{r-1},$$

so that $g = -r \log y + \phi(x)$. Substituting into Eqn. (2.43), we get $\phi'(x) = (1-r)A(x)$, so that $\phi = (1 - r)\int_a^x A(\eta)\, d\eta$. Thus, $g = (1 - r)\int_a^x A(\eta)\, d\eta - r \log y$, and the integrating factor is

$$\mu = e^g = y^{-r} e^{(1-r)\int_a^x A(\eta)\, d\eta}.$$

Now following the procedure in Section 2.4.2, we get

$$\frac{y^{1-r}}{1-r} = \int_a^x e^{(r-1)\int_\xi^x A(\eta)\, d\eta} f(\xi)\, d\xi + ce^{(r-1)\int_a^x A(\eta)\, d\eta}. \tag{2.44}$$

Note that for $r = 0$, the solution reduces to

$$y = \int_a^x e^{-\int_\xi^x A(\eta)\, d\eta} f(\xi)\, d\xi + ce^{-\int_a^x A(\eta)\, d\eta},$$

which is the same as the solution in Eqn. (2.18a).

3. A generalized Riccati equation is of the form

$$\left[p(x) + q(x)y + r(x)y^2\right] dx - dy = 0. \tag{2.45}$$

Thus, $M = p(x) + q(x)y + r(x)y^2$ and $N = -1$. By carrying out the first two checks in the above procedure, we see that $g = g(x, y)$ in this case. Thus,

$$g_x + \left[p(x) + q(x)y + r(x)y^2\right] g_y = -q(x) - 2r(x)y. \tag{2.46}$$

The '$q(x)y$' term in Eqn. (2.45) can be eliminated by a change of variable as follows. Let $y = ze^{\int q\,dx}$, so that $y' = z'e^{\int q\,dx} + qy$. Substituting into Eqn. (2.45), we get

$$z'(x) = p(x)e^{-\int q\,dx} + r(x)z^2(x)e^{\int q\,dx}. \tag{2.47}$$

However, this form yields no particular advantage over that of Eqn. (2.45), and so all the remaining discussion pertains to Eqn. (2.45).

If $y_1(x)$ is one solution to the Riccati equation, then the other solution is given by

$$y_2(x) = y_1(x) + \frac{e^{\int_a^x q(\eta)+2r(\eta)y_1(\eta)\,d\eta}}{c - \int_a^x e^{\int_a^\xi q(\eta)+2r(\eta)y_1(\eta)\,d\eta} r(\xi)\,d\xi}, \tag{2.48}$$

where the lower integration limit a can be set to zero if the resulting integrals are well-behaved. To prove Eqn. (2.48), substitute $y_2(x) = y_1(x) + u(x)$ into Eqn. (2.45) to get the governing differential equation for $u(x)$ as

$$u(x)\left[q(x) + r(x)(u(x) + 2y_1(x))\right] - u'(x) = 0. \tag{2.49}$$

But this is just the Bernoulli equation given by Eqn. (2.42) with $r = 2$, $A(x) = -[q(x) + 2y_1(x)r(x)]$ and $f(x) = r(x)$. Thus, the solution obtained using Eqn. (2.44) is

$$u(x) = \frac{e^{\int_a^x [q(\eta)+2r(\eta)y_1(\eta)]\,d\eta}}{c - \int_a^x e^{\int_a^\xi [q(\eta)+2r(\eta)y_1(\eta)]\,d\eta} r(\xi)\,d\xi},$$

and since $y_2(x) = y_1(x) + u(x)$, Eqn. (2.48) follows.

Although it was possible to find an integrating factor in the case of the Bernoulli equation, we see that it may not be possible to find an integrating factor analytically for arbitrary $p(x)$, $q(x)$ and $r(x)$ in Eqn. (2.45). However, under some assumptions, one can find an analytical solution. We now consider various such cases.

- Let

$$q(x) = \frac{p'(x)}{2p(x)} - 2\sqrt{p(x)},$$

$$r(x) = 1.$$

It is obvious that $y = \sqrt{p(x)}$ is a solution to Eqn. (2.45). The other solution is obtained using Eqn. (2.48) as

$$y = \sqrt{p(x)} + \frac{\sqrt{p(x)}}{c - \int_0^x \sqrt{p(\xi)}\, d\xi}.$$

- Let $p(x)$, $q(x)$ and $r(x)$ be such that

$$p(x) + q(x)y + r(x)y^2 = [h(x) + r(x)y][\alpha + y],$$

where α is a *constant*. It is obvious that $y = -\alpha$ is a solution to Eqn. (2.45). The other solution obtained using Eqn. (2.48) is

$$y = -\alpha + \frac{e^{\int_a^x [q(\xi) - 2\alpha r(\xi)]\, d\xi}}{c - \int_a^x e^{\int_a^\xi [q(\eta) - 2\alpha r(\eta)]\, d\eta} r(\xi)\, d\xi}. \tag{2.50}$$

- Let $a(x)$ be some function, and let

$$q(x) = \frac{a(x)p(x)}{a'(x)} + \frac{p'(x)}{p(x)} - \frac{a''(x)}{a'(x)}, \tag{2.51a}$$

$$r(x) = -1, \tag{2.51b}$$

in which case $y_1 = a(x)p(x)/a'(x)$ is one solution of Eqn. (2.45), as can be verified by direct substitution. The other solution is obtained from Eqn. (2.48). Thus, the two solutions of Eqn. (2.45) are

$$y_1(x) = \frac{a(x)p(x)}{a'(x)}, \tag{2.52a}$$

$$y_2(x) = \frac{a(x)p(x)}{a'(x)} + \frac{\frac{p(x)}{a'(x)} e^{-\int_a^x \frac{a(\eta)p(\eta)}{a'(\eta)}\, d\eta}}{\int_a^x \frac{p(\xi)}{a'(\xi)} e^{-\int_a^\xi \frac{a(\eta)p(\eta)}{a'(\eta)}\, d\eta}\, d\xi + c}, \tag{2.52b}$$

where c is a constant.

As examples of the above for various choices of $a(x)$, we have the following:

(a) If the constraint is

$$q(x) = \frac{x^{1-\beta}p(x)}{\alpha\beta} + \frac{p'(x)}{p(x)} - \frac{\alpha\beta x^\beta + \beta - 1}{x},$$

the solutions obtained using Eqns. (2.52) (after renaming the constant) are

$$\begin{aligned} y_1(x) &= \frac{x^{1-\beta}p(x)}{\alpha\beta}, \\ y_2(x) &= \frac{x^{1-\beta}p(x)}{\alpha\beta} + \frac{x^{1-\beta}p(x)e^{-\alpha x^\beta}e^{-\frac{1}{\alpha\beta}\int_a^x \eta^{1-\beta}p(\eta)\,d\eta}}{c+\int_a^x \xi^{1-\beta}p(\xi)e^{-\alpha\xi^\beta}e^{-\frac{1}{\alpha\beta}\int_a^\xi \eta^{1-\beta}p(\eta)\,d\eta}\,d\xi}. \end{aligned}$$

(b) If the constraint is

$$q(x) = \frac{1-\alpha}{x} + \frac{xp(x)}{\alpha} + \frac{p'(x)}{p(x)},$$

the solutions are

$$\begin{aligned} y_1(x) &= \frac{xp(x)}{\alpha}, \\ y_2(x) &= \frac{xp(x)}{\alpha} + \frac{x^{1-\alpha}p(x)e^{-\frac{1}{\alpha}\int_a^x \eta p(\eta)\,d\eta}}{c+\int_a^x \xi^{1-\alpha}p(\xi)e^{-\frac{1}{\alpha}\int_a^\xi \eta p(\eta)\,d\eta}\,d\xi}. \end{aligned}$$

(c) If the constraint is

$$p(x) = q'(x) + \alpha\beta x^{\beta-1}q(x) + \alpha\beta x^{\beta-2}\left[\alpha\beta x^\beta + \beta - 1\right].$$

the solutions are

$$\begin{aligned} y_1(x) &= q(x) + \alpha\beta x^{\beta-1}, \\ y_2(x) &= q(x) + \alpha\beta x^{\beta-1} + \frac{e^{-2\alpha x^\beta}e^{-\int_a^x q(\eta)\,d\eta}}{c+\int_a^x e^{-2\alpha\xi^\beta}e^{-\int_a^\xi q(\eta)\,d\eta}\,d\xi}. \end{aligned}$$

(d) If the constraint is

$$p(x) = -\frac{\alpha}{x^2}\left[xq(x) + 1 - \alpha\right],$$

the solutions are

$$\begin{aligned} y_1(x) &= \frac{\alpha}{x}, \\ y_2(x) &= \frac{\alpha}{x} + \frac{x^{-2\alpha}e^{\int_a^x q(\eta)\,d\eta}}{c+\int_a^x \xi^{-2\alpha}e^{\int_a^\xi q(\eta)\,d\eta}\,d\xi}. \end{aligned}$$

(e) If the constraint is

$$p(x) = \alpha\beta x^{\beta-2}\left[\alpha\beta x^\beta + \beta - 1 - xq(x)\right],$$

the solutions are

$$y_1(x) = \alpha\beta x^{\beta-1},$$
$$y_2(x) = \alpha\beta x^{\beta-1} + \frac{e^{-2\alpha x^{\beta}} e^{\int_a^x q(\eta)\,d\eta}}{c + \int_a^x e^{-2\alpha\xi^{\beta}} e^{\int_a^{\xi} q(\eta)\,d\eta}\,d\xi}.$$

(f) If the constraint is

$$p(x) = q'(x) + \frac{\beta(1 + \beta - xq(x))}{x^2},$$

the solutions are

$$y_1(x) = q(x) - \frac{\beta}{x},$$
$$y_2(x) = q(x) - \frac{\beta}{x} + \frac{x^{2\beta} e^{-\int_a^x q(\eta)\,d\eta}}{c + \int_a^x \xi^{2\beta} e^{-\int_a^{\xi} q(\eta)\,d\eta}\,d\xi}.$$

(g) If the constraint is

$$p(x) = -\frac{\alpha\left[q^2(x) + q'(x) - \alpha\right]}{q^2(x)},$$

the solutions are

$$y_1(x) = \frac{\alpha}{q(x)},$$
$$y_2(x) = \frac{\alpha}{q(x)} + \frac{e^{-\int_a^x \frac{2\alpha - q^2(\eta)}{q(\eta)}\,d\eta}}{c + \int_a^x e^{-\int_a^{\xi} \frac{2\alpha - q^2(\eta)}{q(\eta)}\,d\eta}\,d\xi}.$$

(h) If the constraint is

$$p(x) = \alpha\left[(\alpha - 1)q^2(x) + q'(x)\right],$$

the solutions are

$$y_1(x) = \alpha q(x),$$
$$y_2(x) = \alpha q(x) + \frac{e^{-\int_a^x (2\alpha-1)q(\eta)\,d\eta}}{c + \int_a^x e^{-\int_a^{\xi} (2\alpha-1)q(\eta)\,d\eta}\,d\xi}.$$

(i) If the constraint is

$$p(x) = \frac{2q(x)q'(x) + q''(x)}{q(x)},$$

the solutions are

$$y_1(x) = q(x) + \frac{q'(x)}{q(x)},$$
$$y_2(x) = q(x) + \frac{q'(x)}{q(x)} + \frac{[q(x)]^{-2} e^{-\int_a^x q(\eta)\, d\eta}}{c + \int_a^x [q(\xi)]^{-2} e^{-\int_a^\xi q(\eta)\, d\eta}\, d\xi}.$$

(j) If the constraint is

$$p(x) = -\frac{q(x)q''(x) - 2[q'(x)]^2}{q^2(x)},$$

the solutions are

$$y_1(x) = q(x) - \frac{q'(x)}{q(x)},$$
$$y_2(x) = q(x) - \frac{q'(x)}{q(x)} + \frac{q^2(x) e^{-\int_a^x q(\eta)\, d\eta}}{c + \int_a^x q^2(\xi) e^{-\int_a^\xi q(\eta)\, d\eta}\, d\xi}.$$

- If there exist functions $\phi(x)$ and $g(x)$ such that

$$p(x) = \phi'(x) - \phi^2(x) - [g'(x)]^2 - \frac{\phi(x)g''(x)}{g'(x)}, \tag{2.53a}$$
$$q(x) = 2\phi(x) + \frac{g''(x)}{g'(x)}, \tag{2.53b}$$
$$r(x) = -1. \tag{2.53c}$$

the solutions are given by

$$y_1 = \phi(x) - g'(x) \tan g(x),$$
$$y_2 = \phi(x) + g'(x) \cot g(x).$$

By replacing $g(x)$ by $ig(x)$, we get another set of solutions in terms of $\{\tanh g(x), \coth g(x)\}$. Inverting Eqns. (2.53), we have

$$p(x) = \phi'(x) - \phi^2(x) - [e^{\int (q-2\phi)\, dx}]^2 - \phi(x)[q(x) - 2\phi(x)],$$
$$3[g''(x)]^2 - 2g'(x)g'''(x) - 4[g'(x)]^4 = [4p(x) + q^2(x)][g'(x)]^2,$$

which in principle can be solved for $\phi(x)$ and $g(x)$, given $p(x)$ and $q(x)$ (although this is much tougher than solving the original problem!).

As an example, if $p(x)$ and $q(x)$ have the forms $p(x) = \phi'(x) - \phi^2(x)$, $q(x) = 2\phi(x)$ (corresponding to $g =$ constant), then one solution is given by $y_1 = \phi(x)$. The other solution is found using the method of reduction of order to be $1/(x+c)$, where c is a constant. Similarly, if $p(x) = -[g'(x)]^2$ and $q(x) = g''(x)/g'(x)$ (corresponding to $\phi = 0$), then the solutions are $y_1 = -g'(x)\tan g(x)$ and $y_2 = g'(x)\cot g(x)$.

As another example, the solutions of the differential equation

$$y' = \phi'(x) - \phi^2(x) - \alpha^2 + 2\phi(x)y - y^2$$

obtained by taking $g(x) = \alpha x$ are

$$\begin{aligned} y_1 &= \phi(x) - \alpha \tan \alpha x, \\ y_2 &= \phi(x) + \alpha \cot \alpha x. \end{aligned}$$

4. Consider the equation

$$-y\,dx + (x + x^6)\,dy = 0.$$

We have $M = -y$ and $N = x + x^6$. Thus,

$$(x + x^6)g_x + y g_y = -2 - 6x^5. \tag{2.54}$$

Equating $y g_y = -2$, we get $g = -2\log y + \phi(x)$. Substituting into Eqn. (2.54), we get

$$(1 + x^5)\phi'(x) = -6x^4,$$

which yields $\phi(x) = -6\log(1+x^5)/5$. Thus, $g = \log[(1+x^5)^{-6/5}y^{-2}]$ so that $\mu = e^g = (1+x^5)^{-6/5}y^{-2}$ (Verify that $g = \log(y^4/x^6)$ leading to $\mu = y^4/x^6$ is also an integrating factor, which is obtained by writing Eqn. (2.54) as $(x+x^6)g_x + y g_y = 4 - 6(1+x^5)$, and equating $(x+x^6)g_x$ to $-6(1+x^5)$. Although this integrating factor may look different from what we have already derived, it is actually the same (modulo a constant) in light of the final solution given by Eqn. (2.55), although we do not know it at this stage since this solution is an unknown; this shows that the integrating factor can be written in different ways while trying to solve a problem). Thus, the exact equation is

$$-\frac{dx}{y(1+x^5)^{6/5}} + \frac{x\,dy}{y^2(1+x^5)^{1/5}} = 0,$$

which leads to

$$y = \frac{cx}{(1+x^5)^{1/5}}. \tag{2.55}$$

5. Solve
$$(3xy + 6y^2)\,dx + (2x^2 + 9xy)\,dy = 0.$$
Since $M = 3xy + 6y^2$ and $N = 2x^2 + 9xy$, from Eqn. (2.36) we get
$$(2x^2 + 9xy)g_x - 3y(x + 2y)g_y = 3y - x = -3x - 6y + 2x + 9y. \tag{2.56}$$
Equating $-3y(x + 2y)g_y = -3x - 6y$, we get $g = \log y + \phi(x)$, which on substituting into Eqn. (2.56) yields $\phi = \log x$. Thus, $g = \log(xy)$, which leads to $\mu = xy$. Thus, the final solution is given by
$$x^3y^2 + 3x^2y^3 = c.$$

2.5 Second order equations

2.5.1 Nonlinear second order equations

A general second order equation is of the form
$$y'' = f(x, y, y').$$
The equation is said to be linear if f is linear in the arguments y and y'. As is only to be expected, it is difficult to solve nonlinear second order equations. However, in a few special cases, it is possible to transform them into first order equations which are easier to solve.

Equations of the form $y'' = f(x, y')$

By substituting $v = y'$, the differential equation is transformed to the first order equation $v' = f(x, v)$. As an example, consider the second order nonlinear equation
$$y'' = -2x(y')^2,$$
with the initial conditions $y(0) = 2$ and $y'(0) = 1$. By substituting $v = y'$, we get
$$v' = -2xv^2,$$
which can be written as $dv/v^2 = -2x\,dx$, so that
$$-\frac{1}{v} = -x^2 + c_1.$$
Substituting for v, we get
$$\frac{1}{y'} = x^2 - c_1,$$

i.e.,

$$y' = \frac{1}{x^2 - c_1}.$$

Using the initial condition $y'(0) = 1$, we get $c_1 = -1$. Integrating the resulting equation, we get

$$y = \tan^{-1} x + c_2.$$

Using $y(0) = 2$, we get $c_2 = 2$. Thus, $y = \tan^{-1} x + 2$.

As another example, consider the solution of

$$xy'' + 4y' = x^2.$$

Put $y' = v(x)$, so that the equation becomes $v' + 4v/x = x$. This is just a first order linear equation with variable coefficients whose solution is given by $v = y' = x^2/6 + c_1 x^{-4}$. Integrating this once again, and redefining the constants, we get $y = x^3/18 + c_1 x^{-3} + c_2$.

Equations of the form $y'' = f(y, y')$

In this case, make the substitution $v(x) = y'(x)$ so that $v'(x) = f(y, v(x))$. If y is an invertible function of x, then we can write $x = x(y)$, so that $v(x) = v(x(y)) =: u(y)$. Thus,

$$v'(x) = \frac{dv}{dy}\frac{dy}{dx} = u(y)\frac{du}{dy} = f(y, u(y)).$$

which can be written as

$$\frac{du}{dy} = \frac{1}{u} f(y, u(y)). \tag{2.57}$$

As an example, consider the differential equation $y'' = 2yy'$ subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$. If we write $y' = v(x) = u(y)$, then the governing equation obtained using Eqn. (2.57) is

$$\frac{du}{dy} = \frac{1}{u}(2yu) = 2y.$$

Thus, $u = y^2 + c_1$, or, $y' = y^2 + c_1$. Using the initial conditions, we get $c_1 = 1$. Integrating $y' = y^2 + 1$, we get $\tan^{-1} y = x + c_2$, or, $y = \tan(x + c_2)$. Since $y(0) = 0$, we get $c_2 = 0$. Thus, the final solution is $y = \tan x$.

In view of the difficulty in solving nonlinear second order equations, we shall henceforth restrict ourselves to *linear* second order equations.

2.5.2 Linear second order equations

Before we begin this topic, we briefly discuss the linear independence of functions. The set of functions $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ is said to be linearly independent if the equation

$$c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0, \tag{2.58}$$

implies that $c_1 = c_2 = \ldots = c_n = 0$. Equivalently, the set of functions $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ is said to be linearly dependent if there exist constants c_1, c_2, ..., c_n not all zero such that Eqn. (2.58) holds. Let the Wronskian $W(x)$ be given by

$$W(x) := \det \begin{bmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{bmatrix}.$$

where the superscript $n-1$ denotes the $n-1$'th derivative. We have the following theorem:

Theorem 2.5.1. *The set of functions $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ is linearly independent if and only if $W(x) \neq 0$.*

Proof. We prove the equivalent statement that the set of functions $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ is linearly dependent if and only if $W(x) = 0$. By repeatedly differentiating Eqn. (2.58), we get

$$\begin{aligned} c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) &= 0, \\ c_1 y_1'(x) + c_2 y_2'(x) + \ldots + c_n y_n'(x) &= 0, \\ \ldots &= 0, \\ c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \ldots + c_n y_n^{(n-1)}(x) &= 0, \end{aligned}$$

which can be written in the form

$$\begin{bmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ c_n \end{bmatrix} = 0. \tag{2.59}$$

The above matrix equation has a nontrivial solution for $\{c_1, c_2, \ldots, c_n\}$ if and only if $W(x) = 0$, which proves the theorem. □

Note that if $W(x)$ is zero at certain points on the domain, but is not the zero function, the functions are linearly independent. For example, the Wronskian of the functions $y_1 = \cos x$ and $y_2 = \sin^2 x$ is $W(x) = \sin x(1 + \cos^2 x)$. Although the Wronskian is zero at $x = n\pi$, where n is an integer, $W(x)$ is not the zero function, and hence, y_1 and y_2 are linearly independent.

A linear second order equation is of the form

$$a(x)y'' + b(x)y' + c(x)y = f(x). \tag{2.60}$$

Let $\{y_1, y_2\}$ denote the solutions of the homogeneous form of the above equation, i.e.,

$$y'' + p(x)y' + q(x)y = 0. \tag{2.61}$$

The Wronskian of $\{y_1, y_2\}$ is given by

$$W(x) = y_1 y_2' - y_1' y_2. \tag{2.62}$$

By Theorem 2.5.1, if $W(x) \neq 0$, then the set $\{y_1(x), y_2(x)\}$ is linearly independent; in such a case $\{y_1, y_2\}$ are called *fundamental solutions*. Let $\{y_1(x), y_2(x)\}$ be fundamental solutions. Then differentiating Eqn. (2.62), and noting that y_1 and y_2 are solutions of Eqn. (2.61), we get

$$\begin{aligned} W' &= y_1 y_2'' - y_1'' y_2 \\ &= -y_1(p y_2' + q y_2) + y_2(p y_1' + q y_1) \\ &= -p(y_1 y_2' - y_2 y_1') \\ &= -pW. \end{aligned}$$

Solving the above equation, we get

$$W = W(x_0) e^{-\int_a^x p(\xi)\, d\xi}, \tag{2.63}$$

where x_0 is any point in the domain $[a, b]$. Eqn. (2.63) is known as Abel's formula.

If $\{y_1, y_2\}$ are fundamental solutions of Eqn. (2.61), then by the linearity of the governing differential equation, the *most general solution* to Eqn. (2.61) can be written as

$$y(x) = c_1 y_1 + c_2 y_2,$$

where c_1 and c_2 are constants.

Given the fundamental solutions $\{y_1, y_2\}$, we can find the differential equation of which they are solutions by noting that

$$\det \begin{bmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{bmatrix} = 0, \tag{2.64}$$

since by substituting either $y = y_1$ or $y = y_2$ in the above equation, we see that two columns become identical leading to a zero determinant.

Now we discuss how to solve constant coefficient homogeneous equations.

2.5.3 Constant coefficient homogeneous equations

Consider the constant coefficient homogeneous version of Eqn. (2.60):

$$ay'' + by' + cy = 0. \tag{2.65}$$

To solve this equation, we assume $y = e^{rx}$. Substituting into Eqn. (2.65), we get the *characteristic equation*

$$ar^2 + br + c = 0,$$

whose roots are

$$r = \frac{1}{2a}\left[-b \pm \sqrt{b^2 - 4ac}\right].$$

If

1. $b^2 - 4ac > 0$, the characteristic equation has two distinct real roots.
2. $b^2 - 4ac = 0$, the characteristic equation has a repeated real root.
3. $b^2 - 4ac < 0$, the characteristic equation has complex roots.

We illustrate each of these cases by examples.

The roots r_1 and r_2 are real and distinct

Solve

$$y'' + 6y' + 5y = 0, \quad y(0) = 3,\ y'(0) = -1.$$

The characteristic equation is given by

$$0 = r^2 + 6r + 5 = (r+1)(r+5).$$

The two fundamental solutions (e^{-x}, e^{-5x}) are linearly independent since the Wronskian

$$\det\begin{bmatrix} e^{-x} & e^{-5x} \\ -e^{-x} & -5e^{-5x} \end{bmatrix} \neq 0.$$

Thus, the general solution is $y = c_1 e^{-x} + c_2 e^{-5x}$. Using the initial conditions $y(0) = 3$ and $y'(0) = -1$, we get $c_1 = 7/2$ and $c_2 = -1/2$.

The roots are repeated, i.e., $r_1 = r_2 = \alpha$

Find the general solution of

$$y'' - 2\alpha y' + \alpha^2 y = 0.$$

The characteristic polynomial is $(r-\alpha)^2 = 0$, so that one solution is $y_1 = e^{\alpha x}$. Since the roots $r = \alpha$ is repeated, we have to find the other independent solution separately. We use the method of *reduction of order*, where we look for a solution of the form $y_2 = u(x)y_1$. Substituting this form into the differential equation yields $u''(x) = 0$, or $u(x) = k_1 x + k_2$, so that $y_2 = (k_1 x + k_2)e^{\alpha x}$. Since the coefficient of k_2 is y_1 itself, we see that the other fundamental solution is $y_2 = xe^{\alpha x}$, and the general solution is $y(x) = e^{\alpha x}(c_1 + c_2 x)$.

The roots r_1 **and** r_2 **are complex-valued**

If the roots of the characteristic equation are complex-valued (and hence conjugates of each other), we use the formula $e^{i\theta} = \cos\theta + i\sin\theta$ to find the general solution as follows. Let the differential equation be

$$y'' + 4y' + 13y = 0.$$

The characteristic equation is

$$r^2 + 4r + 13 = 0,$$

so that $r_1 = -2+3i$ and $r_2 = -2-3i$ are the roots. Thus, $y_1 = e^{(-2+3i)x}$ and $y_2 = e^{(-2-3i)x}$ are the solutions, which we can write as $y_1 = e^{-2x}(\cos 3x + i\sin 3x)$ and $y_2 = e^{-2x}(\cos 3x - i\sin 3x)$. Thus, one can either write the general solution as $y(x) = c_1 e^{(-2+3i)x} + c_2 e^{(-2-3i)x}$, with the constants c_1 and c_2 *complex-valued* or as $y(x) = e^{-2x}(c_1\cos 3x + c_2 \sin 3x)$, with the constants c_1 and c_2 real-valued. Even if one uses the complex-valued form, on using the initial conditions, the constants will turn out to be complex-conjugates of each other, so that the final result is real-valued. For example, if the initial conditions are $y(0) = 2$ and $y'(0) = -3$, then we get $c_1 = 1 - i/6$ and $c_2 = 1 + i/6$, so that

$$y(x) = e^{-2x}\left(2\cos 3x + \frac{1}{3}\sin 3x\right).$$

2.5.4 Nonhomogeneous linear equations

Consider Eqn. (2.60), which we now write in the form (by redefining $f(x)$ to be $f(x)/a(x)$)

$$y'' + p(x)y' + q(x)y = f(x). \tag{2.66}$$

The associated homogeneous equation is

$$y'' + p(x)y' + q(x)y = 0. \tag{2.67}$$

Eqn. (2.67) is known as the complementary equation for Eqn. (2.66). Let y_p be a particular solution of Eqn. (2.66), and let $\{y_1, y_2\}$ be fundamental solutions of the complementary equation given by Eqn. (2.67). Then, by the linearity of the governing differential equation, the general solution to Eqn. (2.66) is the sum of the complementary and the particular solutions, i.e.,

$$y = c_1 y_1 + c_2 y_2 + y_p. \tag{2.68}$$

At this stage, we have not discussed how to find y_p, or even how to find $\{y_1, y_2\}$ if the coefficients $p(x)$ and $q(x)$ are functions of x (in the previous section, we have seen how to find $\{y_1, y_2\}$ in case $p(x)$ and $q(x)$ are constants); we will do this at a later stage. After the fundamental solution set $\{y_1, y_2\}$ is found, the particular solution y_p is found using the method of variation of parameters as described in Section 2.5.6.

2.5.5 Finding the second complementary solution given the first complementary solution using the method of reduction of order

If one complementary solution to Eqn. (2.67), say $y_1(x)$, is known, then the other complementary solution can be found using the method of reduction of order, whereby we assume the second solution to be given by $y_2 = u(x)y_1(x)$ and substitute into the governing differential equation given by Eqn. (2.67) to get

$$\frac{u''}{u'} = -\frac{p(x)y_1(x) + 2y_1'(x)}{y_1(x)},$$

which can be easily integrated twice to obtain

$$y_2 = y_1(x) \int_a^x \frac{e^{-\int_a^\xi p(\eta)\, d\eta}}{y_1^2(\xi)}\, d\xi. \tag{2.69}$$

The lower limit of integration a can be taken to be zero if the integrals are well-behaved.

2.5.6 Finding the particular solution y_p using the method of variation of parameters

Given the fundamental solutions $\{y_1, y_2\}$, we now discuss a systematic procedure for finding the general solution (which essentially means finding the particular solution since the complementary solution is known, and given by $y_c = c_1 y_1 + c_2 y_2$) to the inhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x). \tag{2.70}$$

Let the general solution be given by

$$y = u_1 y_1 + u_2 y_2, \tag{2.71}$$

where u_1 and u_2 are both functions of x. By substituting Eqn. (2.71) into Eqn. (2.70), we obtain one condition on u_1 and u_2. However, we need two conditions. The second condition is obtained as follows. Differentiating Eqn. (2.71), we get

$$y' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2. \tag{2.72}$$

We require that

$$u_1' y_1 + u_2' y_2 = 0, \tag{2.73}$$

so that from Eqn. (2.72), we have $y' = u_1 y_1' + u_2 y_2'$ and $y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$. Substituting these expressions into Eqn. (2.70), and noting that y_1 and y_2 are complementary solutions, we get

$$u_1' y_1' + u_2' y_2' = f(x). \tag{2.74}$$

Solving for u_1' and u_2' using Eqns. (2.73) and (2.74), we get

$$u_1' = -\frac{f y_2}{y_1 y_2' - y_1' y_2}, \tag{2.75a}$$

$$u_2' = \frac{f y_1}{y_1 y_2' - y_1' y_2}. \tag{2.75b}$$

Note that since $\{y_1, y_2\}$ are linearly independent, the denominator in Eqns. (2.75) (which is the same as the Wronskian) is nonzero. Thus, the final solution is

$$u_1 = -\int \frac{f y_2 \, dx}{y_1 y_2' - y_1' y_2} + c_1, \tag{2.76a}$$

$$u_2 = \int \frac{f y_1 \, dx}{y_1 y_2' - y_1' y_2} + c_2, \tag{2.76b}$$

where c_1 and c_2 are constants of integration. By substituting the above expressions into Eqn. (2.71), we obtain the most general solution to Eqn. (2.70). Note that the part of the solution associated with the constants c_1 and c_2 is merely the complementary solution, while the remaining part is the particular solution y_p that we intended to find.

As an example, given that $(y_1, y_2) = (x, x^2)$, find the solution to

$$x^2 y'' - 2xy' + 2y = x^{9/2},$$

which we write as

$$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = x^{5/2}.$$

We write the general solution as

$$y = u_1(x)x + u_2(x)x^2,$$

where

$$\begin{aligned} u_1'x + u_2'x^2 &= 0, \\ u_1' + 2u_2'x &= x^{5/2}. \end{aligned}$$

Solving for u_1' and u_2', we get

$$\begin{aligned} u_1' &= -x^{5/2}, \\ u_2' &= x^{3/2}, \end{aligned}$$

which on integrating yield

$$\begin{aligned} u_1 &= -\frac{2}{7}x^{7/2} + c_1, \\ u_2 &= \frac{2}{5}x^{5/2} + c_2. \end{aligned}$$

Thus, the general solution is

$$y = c_1x + c_2x^2 - \frac{2}{7}x^{9/2} + \frac{2}{5}x^{9/2} = c_1x + c_2x^2 + \frac{4}{35}x^{9/2}. \tag{2.77}$$

As another example, consider finding the general solution to

$$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = x - 1.$$

given that $y_1 = x$ and $y_2 = e^x$.

The general solution is written as

$$y = u_1(x)x + u_2(x)e^x,$$

where u_1 and u_2 satisfy

$$\begin{aligned} u_1'x + u_2'e^x &= 0, \\ u_1' + u_2'e^x &= x - 1. \end{aligned}$$

Solving for u_1' and u_2', we get $u_1' = -1$ and $u_2' = xe^{-x}$. Integrating, we get $u_1 = -x + c_1$ and $u_2 = -(1+x)e^{-x} + c_2$. Thus, the general solution is

$$y = (c_1 - 1)x + c_2e^x - x^2 - 1,$$

which, by redefining c_1 can be rewritten as

$$y = c_1x + c_2e^x - x^2 - 1.$$

The constants c_1 and c_2 are found as usual by using the initial conditions.

2.5.7 Second order equations with variable coefficients

We will consider only the homogeneous case here, since once the complementary solution is found, the particular solution can be found using the method of variation of parameters as discussed in the previous subsection. First note that a second order equation with variable coefficients can be converted to a Riccati equation and vice versa, so that known Riccati solutions can be used to solve the current problem and vice versa. To see this, we write $y = e^{g(x)}$ so that

$$\begin{aligned} y' &= e^g g', \\ y'' &= e^g \left[g'' + (g')^2 \right], \end{aligned}$$

where, as usual, primes denote derivatives with respect to x. Substituting the above relations into Eqn. (2.81), we get

$$g'' + (g')^2 + pg' + q = 0.$$

Substituting $g' = u$, we get

$$u' + u^2 + pu + q = 0,$$

which is nothing but the Riccati equation given by Eqn. (2.45) with $r = -1$, and with p denoted by $-q$, and q denoted by $-p$. Conversely, the Riccati equation given by Eqn. (2.45) with $r = -1$ can be transformed into a second order equation with variable coefficients using the substitution $y = z'/z$.

Now we discuss techniques for solving variable coefficient second order equations. First consider the solution of the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{2.78}$$

Similar to the case of first order equations, the above equation is said to be exact if

$$P''(x) - Q'(x) + R(x) = 0. \tag{2.79}$$

If an equation is exact, then Eqn. (2.78) can be written as

$$[P(x)y']' + [(Q(x) - P'(x))y]' = 0, \tag{2.80}$$

which on integrating yields

$$P(x)y' + [Q(x) - P'(x)]y = c_1.$$

The above first order linear differential equation can be solved by standard techniques.

As an example, consider the solution of the equation

$$x(x-2)y'' + 4(x-1)y' + 2y = 0.$$

Since Eqn. (2.79) is satisfied, the equation is exact. Eqn. (2.80) reduces to

$$x(x-2)y' + 2(x-1)y = c_1,$$

which on integrating yields $x(x-2)y = c_1x + c_2$, so that the solution is

$$y = \frac{c_1x + c_2}{x(x-2)},$$

In case the equation is not exact, finding an integration factor is, in general, difficult, and we look for other techniques for solving Eqn. (2.78). By dividing by $P(x)$, we rewrite Eqn. (2.78) in the form

$$y'' + p(x)y' + q(x)y = 0. \tag{2.81}$$

In general, finding solutions to Eqn. (2.81) is difficult. Sometimes it is convenient to transform Eqn. (2.81) into the standard form

$$z''(x) + v(x)z = 0, \tag{2.82}$$

where

$$\log y = \log z - \frac{1}{2}\int p(x)\,dx, \tag{2.83a}$$

$$v(x) = q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p^2(x). \tag{2.83b}$$

The techniques that we describe below can be applied to either Eqn. (2.81) or Eqn. (2.82).

One special case of variable coefficients where the solution can be easily found is when $p(x) = \alpha/x$ and $q(x) = \beta/x^2$, where α and β are constants, in which case the equation is known as the *Euler equation*. The solution to the homogeneous equation given by Eqn. (2.81) is found by substituting $y = x^r$ to get the characteristic equation as

$$r^2 + (\alpha - 1)r + \beta = 0.$$

As in the constant coefficient case, we need to consider three possibilities.

The roots r_1 and r_2 are real and distinct

In this case $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$ are fundamental solutions, and the general complementary solution is $y_c = c_1x^{r_1} + c_2x^{r_2}$.

The roots are repeated, i.e., $r_1 = r_2$

The roots are real and repeated if $4\beta = (\alpha - 1)^2$, and are given by $r_1 = r_2 = (1 - \alpha)/2$. In this case, one of the solutions is $y_1 = x^{r_1}$. In order to find the other solution, we use the method of variation of parameters and write the other solution as $y_2 = u(x)x^{r_1}$. Substituting into $y'' + (\alpha/x)y' + (\beta/x^2)y = 0$, we get $xu''(x) + u'(x) = 0$, whose solution is $u(x) = k_1 \log x + k_2$, so that $y_2 = (k_1 \log x + k_2)x^r$. The coefficient of k_2 is y_1 itself, so that the other complementary solution is $y_2 = x^{r_1} \log x$, and the general complementary solution for this case is $y_c = x^{r_1}(c_1 + c_2 \log x)$.

The roots r_1 and r_2 are complex-valued

In this case, the general solution can be written either as $y_c = c_1 x^{r_1} + c_2 x^{r_2}$, where c_1 and c_2 are complex-valued constants, or, if one wants to deal only with real-valued constants, as $y_c = x^a[c_1 \cos(b \log x) + c_2 \sin(b \log x)]$, where we have used $x^{ib} = e^{ib \log x} = \cos(b \log x) + i \sin(b \log x)$.

An alternative way to derive the same result is as follows. If we consider the Euler equation in the form

$$ax^2y'' + bxy' + cy = 0, \tag{2.84}$$

where a, b and c are constants, and make the transformation $\xi = \log x$, then we see that Eqn. (2.81) is transformed to a second order equation with *constant* coefficients; this follows by noting that

$$\begin{aligned} y' &= \frac{1}{x}\frac{dy}{d\xi}, \\ y'' &= -\frac{1}{x^2}\frac{dy}{d\xi} + \frac{1}{x^2}\frac{d^2y}{d\xi^2}. \end{aligned}$$

Now we try and see the conditions under which Eqn. (2.81) can be transformed to a second order equation with constant coefficients. Since the coefficient of y has to be constant, then by dividing by $q(x)$, we write Eqn. (2.81) as

$$\frac{1}{q(x)}y'' + \frac{p(x)}{q(x)}y' + y = 0. \tag{2.85}$$

We look for a transformation $\xi = g(x)$ that will transform Eqn. (2.85) into an equation with constant coefficients (if such a transformation cannot be found, then Eqn. (2.85) *cannot* be transformed into a second order differential equation with constant coefficients, and some other solution method has to be found). Substituting

$$y' = \frac{dy}{d\xi}g',$$

$$y'' = \frac{dy}{d\xi}g'' + \frac{d^2y}{d\xi^2}(g')^2,$$

where primes denote derivatives with respect to x, into Eqn. (2.85), we get

$$\frac{(g')^2}{q}\frac{d^2y}{d\xi^2} + \frac{g'' + pg'}{q}\frac{dy}{d\xi} + y = 0. \tag{2.86}$$

We require that

$$(g')^2 = c_1 q, \tag{2.87a}$$
$$g'' + pg' = c_2 q, \tag{2.87b}$$

where c_1 and c_2 are constant. From Eqn. (2.87a) we have $g' = \sqrt{c_1 q}$, which on substituting into Eqn. (2.87b) yields

$$\frac{\sqrt{c_1}[2p(x)q(x) + q'(x)]}{2[q(x)]^{3/2}} = c_2.$$

Thus, if

$$\frac{2p(x)q(x) + q'(x)}{[q(x)]^{3/2}} = c, \tag{2.88}$$

where c is a constant, then Eqn. (2.81) can be transformed into an equation with constant coefficients. From Eqns. (2.86) and (2.87a), we see that the solution has to be of the form $e^{r\xi} = e^{rg(x)} = e^{\int_h^x \alpha\sqrt{q(\xi)}\,d\xi}$, where α is a constant. If we take the solutions to be

$$y_1 = e^{\int_h^x \alpha\sqrt{q(\xi)}\,d\xi}, \tag{2.89a}$$
$$y_2 = e^{\int_h^x \sqrt{q(\xi)}/\alpha\,d\xi}, \tag{2.89b}$$

substitute them into Eqn. (2.81) and use Eqn. (2.88), we get

$$\frac{2(1+\alpha^2)}{\alpha} = -c. \tag{2.90}$$

The solution procedure may thus be summarized as follows. Check if the left hand side of Eqn. (2.88) is a constant. If it is not a constant, then some other solution method has to be sought. If it is constant, say, c, then solve Eqn. (2.90) for α, and then the solutions are given by Eqns. (2.89) (with any one of the two roots for α). If $c = -4$, then $\alpha = 1$ is a repeated root, while if $c = 4$, then $\alpha = -1$ is a repeated root; in these cases, one has to use the procedure of reduction of order to find the other solution. For both these cases (i.e., $|c| = 4$), the solutions can be written as

$$\left.\begin{aligned} y_1 &= e^{-\frac{c}{4}\int_h^x \sqrt{q(\xi)}\,d\xi}, \\ y_2 &= e^{-\frac{c}{4}\int_h^x \sqrt{q(\xi)}\,d\xi} \int_h^x e^{-\int_h^\xi \left[p(\eta) - \frac{c}{2}\sqrt{q(\eta)}\right]d\eta}\,d\xi. \end{aligned}\right\} \qquad (|c| = 4).$$

It is sometimes advantageous to work directly with Eqn. (2.81), and sometimes with its standard form given by Eqn. (2.82) as the following examples show.

1. Solve
$$4x^2y'' - 4xy' + (4x^2 + 3)y = 0.$$
Since $p(x) = -1/x$ and $q(x) = (4x^2 + 3)/(4x^2)$, from Eqn. (2.88) we see that the left hand side is not a constant. Thus, it is not possible to transform this equation into one with constant coefficients, and hence it is not evident how to solve this particular equation. However, if we convert this equation to standard form, we get from Eqns. (2.82) and (2.83)
$$z''(x) + z(x) = 0,$$
$$y = \sqrt{x}z,$$
Thus, $z_1 = \cos x$, $z_2 = \sin x$, and hence $y_1 = \sqrt{x}\cos x$ and $y_2 = \sqrt{x}\sin x$ are the solutions to the above problem!
As another example, consider finding the solutions to
$$y''(x) + \frac{2(x^2 - 1)}{x^3}y'(x) + \frac{x^2 + 1}{x^6}y(x) = 0. \tag{2.91}$$
On converting the above equation to standard form, we get
$$z''(x) = 0,$$
$$y(x) = \frac{z}{x}e^{-1/(2x^2)},$$
so that the complementary solutions are
$$y_1 = e^{-1/(2x^2)}, \tag{2.92a}$$
$$y_2 = \frac{1}{x}e^{-1/(2x^2)}. \tag{2.92b}$$
The dramatic advantage that working with the standard form can sometimes have is evident from the above examples.

2. This example, on the other hand, shows that in some other cases working with the original form is better than transforming it to standard form. Solve
$$4xy'' + 2y' + y = 0.$$
Since $p(x) = 1/(2x)$ and $q(x) = 1/(4x)$, from Eqn. (2.88) we see that the left hand side is a constant, while for the standard form it is not. Thus, it is advantageous to work

with the original form itself in this case. From Eqn. (2.87a), we get $\xi = g(x) = \sqrt{x}$, while from Eqn. (2.86), we get

$$\frac{d^2y}{d\xi^2} + y = 0,$$

which leads to the solution

$$\begin{aligned} y_c &= c_1 \cos\xi + c_2 \sin\xi \\ &= \sin\sqrt{x} + \cos\sqrt{x}. \end{aligned}$$

3. For the Euler equation given by Eqn. (2.84), we have $p(x) = b/(ax)$ and $q(x) = c/(ax^2)$, from which we see that the left hand side of Eqn. (2.88) is a constant, so that the equation can be transformed into one with constant coefficients. From Eqn. (2.87a), we get $\xi = g(x) = \log x$. We have already seen how to solve this equation.

4. This example shows that in some cases, it may not be possible to transform either the original or the standard form into an equation with constant coefficients. Solve

$$(x-1)y'' - xy' + y = 0. \tag{2.93}$$

Here we have $p(x) = -x/(x-1)$ and $q(x) = 1/(x-1)$. The left hand side of Eqn. (2.88) is not a constant, and hence one has to use some other method as shown below.

One other method that works quite well is as follows. Let one of the solutions of Eqn. (2.81) be $y_1 = x^\alpha$, where α is a constant. Then substituting this solution into Eqn. (2.69), we get the two solutions as

$$\begin{aligned} y_1(x) &= x^\alpha, \\ y_2(x) &= x^\alpha \int_a^x \frac{e^{-\int_a^\xi p(\eta)\,d\eta}}{\xi^{2\alpha}}\,d\xi. \end{aligned} \tag{2.94}$$

Now if we substitute these solutions into Eqn. (2.64), we get

$$q(x) = \frac{\alpha}{x^2}\left[1 - \alpha - xp(x)\right]. \tag{2.95}$$

Thus, if $q(x)$ is of the above form, then the solutions given by Eqns. (2.94) are the two complementary solutions! As an example, consider finding the solution of

$$(1 - x\cot x)y'' - xy' + y = 0.$$

The constraint given by Eqn. (2.95) is satisfied with $\alpha = 1$. Thus, $y_1 = x$ and $y_2(x) = \sin x$ are the complementary solutions. In an analogous manner,

- If

$$q(x) = -\alpha\beta x^{\beta-2}\left[xp(x) + \alpha\beta x^{\beta} + \beta - 1\right], \tag{2.96}$$

then

$$y_1(x) = e^{\alpha x^{\beta}}, \tag{2.97a}$$

$$y_2(x) = e^{\alpha x^{\beta}} \int_a^x \frac{e^{-\int_a^{\xi} p(\eta)\, d\eta}}{e^{2\alpha\xi^{\beta}}}\, d\xi. \tag{2.97b}$$

The coefficients in Eqn. (2.93) satisfy the constraint on q given by Eqn. (2.96) with $\alpha = \beta = 1$, and hence $y_1 = e^x$ and $y_2 = x$ are the complementary solutions. As another example, consider the differential equation

$$y''(x) + xy'(x) + y(x) = 0. \tag{2.98}$$

The constraint given by Eqn. (2.96) is satisfied with $\alpha = -1/2$ and $\beta = 2$. Thus, the complementary solutions as given by Eqns. (2.97) are

$$y_1(x) = e^{-x^2/2}, \tag{2.99a}$$

$$y_2(x) = e^{-x^2/2} \int_0^x e^{\xi^2/2}\, d\xi. \tag{2.99b}$$

As yet another example, consider finding the solutions to Eqn. (2.91), namely,

$$y''(x) + \frac{2(x^2-1)}{x^3} y'(x) + \frac{x^2+1}{x^6} y(x) = 0.$$

The constraint given by Eqn. (2.96) is satisfied with $\alpha = -1/2$ and $\beta = -2$. Thus, the solutions obtained using Eqn. (2.97) are the same as those given by Eqns. (2.92).

- If

$$q(x) = \alpha\left[\alpha - p(x)\cot\alpha x\right],$$

then

$$y_1(x) = \sin\alpha x,$$

$$y_2(x) = \sin\alpha x \int_a^x \frac{e^{-\int_a^{\xi} p(\eta)\, d\eta}}{\sin^2\alpha\xi}\, d\xi.$$

and if

$$q(x) = \alpha\left[\alpha + p(x)\tan\alpha x\right],$$

then

$$y_1(x) = \cos\alpha x,$$

$$y_2(x) = \cos\alpha x \int_a^x \frac{e^{-\int_a^{\xi} p(\eta)\, d\eta}}{\cos^2\alpha\xi}\, d\xi.$$

- If
$$q(x) = -2\alpha\left[p(x)\csc 2\alpha x + \alpha\sec^2\alpha x\right],$$
then
$$y_1(x) = \tan\alpha x,$$
$$y_2(x) = \tan\alpha x\int_a^x \frac{e^{-\int_a^\xi p(\eta)\,d\eta}}{\tan^2\alpha\xi}\,d\xi.$$
and if
$$q(x) = 2\alpha\left[p(x)\csc 2\alpha x - \alpha\csc^2\alpha x\right],$$
then
$$y_1(x) = \cot\alpha x,$$
$$y_2(x) = \cot\alpha x\int_a^x \frac{e^{-\int_a^\xi p(\eta)\,d\eta}}{\cot^2\alpha\xi}\,d\xi.$$

- If
$$q(x) = \frac{1 - xp(x)}{x^2\log\alpha x},$$
then
$$y_1(x) = \log\alpha x,$$
$$y_2(x) = \log\alpha x\int_a^x \frac{e^{-\int_a^\xi p(\eta)\,d\eta}}{\log^2\alpha\xi}\,d\xi.$$

If there exist functions $\phi(x)$ and $g(x)$ such that

$$p(x) = -\left[\frac{2\phi'(x)}{\phi(x)} + \frac{g''(x)}{g'(x)}\right], \tag{2.100a}$$
$$q(x) = [g'(x)]^2 + \frac{\phi'(x)g''(x)}{\phi(x)g'(x)} + \left(\frac{\phi'(x)}{\phi(x)}\right)^2 - \left(\frac{\phi'(x)}{\phi(x)}\right)', \tag{2.100b}$$

then the complementary solutions to Eqn. (2.81) are given by

$$y_1 = \phi(x)\cos g(x), \tag{2.101a}$$
$$y_2 = \phi(x)\sin g(x). \tag{2.101b}$$

By replacing $g(x)$ by $ig(x)$ in all the above equations, we get complementary solutions in terms of $\cosh g(x)$ and $\sinh g(x)$. By inverting Eqns. (2.100), we get

$$q(x) = [g'(x)]^2 + \frac{1}{4}\left[p^2(x) - \left(\frac{g''(x)}{g'(x)}\right)^2\right] + \frac{1}{2}\left[p(x) + \frac{g''(x)}{g'(x)}\right]',$$
$$\frac{\phi'(x)}{\phi(x)} = -\frac{1}{2}\left[p(x) + \frac{g''(x)}{g'(x)}\right],$$

which in principle can be used to find $g(x)$ and $\phi(x)$, given $p(x)$ and $q(x)$ (although, of course, this is a tougher problem than solving the original differential equation!). If $p(x) = 0$ and $q(x) = \alpha^2$, where α is a constant, then, as expected, we get $g(x) = \alpha x$ and $\phi(x) = A$, where A is a constant.

As an example, if $p(x)$ and $q(x)$ are given by Eqns. (2.100) with $g(x) = \alpha x$, i.e.,

$$p(x) = -\frac{2\phi'(x)}{\phi(x)},$$
$$q(x) = \alpha^2 + \left(\frac{\phi'(x)}{\phi(x)}\right)^2 - \left(\frac{\phi'(x)}{\phi(x)}\right)' = \alpha^2 + \frac{p^2(x)}{4} + \frac{p'(x)}{2},$$

then $y_1 = \phi(x)\cos\alpha x$ and $y_2 = \phi(x)\sin\alpha x$ are the complementary solutions. For the case $\alpha = 0$, $y_1 = \phi(x)$ and $y_2 = x\phi(x)$ are the complementary solutions. By putting $\phi = x^n$, we see that the general solution of the differential equation

$$y'' - \frac{2n}{x} + \left[\alpha^2 + \frac{n(n+1)}{x^2}\right] y = 0,$$

is

$$y = x^n(c_1 \sin\alpha x + c_2 \cos\alpha x).$$

For $n = -1/2$, the above general solution is equivalent to $c_1 J_{-1/2}(x) + c_2 Y_{-1/2}(x)$, as expected.

Yet another method that can be attempted is the following. We write Eqn. (2.81) as

$$\boldsymbol{v}' + \boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}.$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ q & p \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} y \\ y' \end{bmatrix}.$$

Similar to the concept of an integrating factor, we write the above equation as

$$(\boldsymbol{P}\boldsymbol{v})' = \boldsymbol{0}, \tag{2.102}$$

which on expanding can be written as

$$\boldsymbol{P}\boldsymbol{v}' + \boldsymbol{P}'\boldsymbol{v} = \boldsymbol{0},$$

or, alternatively,

$$\boldsymbol{v}' + \boldsymbol{P}^{-1}\boldsymbol{P}'\boldsymbol{v} = \boldsymbol{0}.$$

Thus, we need $\boldsymbol{P}^{-1}\boldsymbol{P} = \boldsymbol{A}$. After some calculation, we get

$$\boldsymbol{P} = \begin{bmatrix} a(x) & -\frac{a'(x)[b(x)a'(x)-a(x)b'(x)]}{a'(x)b''(x)-b'(x)a''(x)} \\ b(x) & -\frac{b'(x)[b(x)a'(x)-a(x)b'(x)]}{a'(x)b''(x)-b'(x)a''(x)}. \end{bmatrix} = \begin{bmatrix} a(x) & \frac{a'(x)}{q(x)} \\ b(x) & \frac{b'(x)}{q(x)} \end{bmatrix},$$

with

$$p(x) = \frac{b(x)[a'(x)/q(x)]' - a(x)[b'(x)/q(x)]'}{b(x)[a'(x)/q(x)] - a(x)[b'(x)/q(x)]}, \tag{2.103a}$$

$$q(x) = \frac{b'(x)a''(x) - a'(x)b''(x)}{b(x)a'(x) - a(x)b'(x)}, \tag{2.103b}$$

$$p(x)q(x) + q'(x) = \frac{[b(x)a''(x) - a(x)b''(x)]\,[b'(x)a''(x) - a'(x)b''(x)]}{(b(x)a'(x) - a(x)b'(x))^2}. \tag{2.103c}$$

The solution to Eqn. (2.102) is given by $\boldsymbol{P}\boldsymbol{v} = \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector, so that $\boldsymbol{v} = \boldsymbol{P}^{-1}\boldsymbol{c}$. Thus, we get the complementary solutions as

$$y_1 = \frac{b'(x)}{b(x)a'(x) - a(x)b'(x)}, \tag{2.104a}$$

$$y_2 = \frac{a'(x)}{b(x)a'(x) - a(x)b'(x)}. \tag{2.104b}$$

Given $p(x)$ and $q(x)$, one would ideally like to solve for $a(x)$ and $b(x)$ using Eqns. (2.103), and then the solution follows from Eqns. (2.104). However, since these are much more difficult to solve compared to the original problem, we make specific choices for $a(x)$ or $b(x)$ which imposes constraints on $p(x)$ and $q(x)$. One choice is $a(x) = k_0/b(x)$, where k_0 is a constant. However, this just leads to the same constraint as in Eqn. (2.88). Hence, we now try and simplify these equations by eliminating one of the functions, say $b(x)$, to get a relation totally in terms of $a(x)$ (see Eqn. (2.106) below).

Let $D_0 := b(x)a'(x) - a(x)b'(x)$ denote the denominator of $q(x)$. Note that

$$\frac{[a(x)D_0/a'(x)]'}{[a(x)D_0/a'(x)]} = \frac{a'(x)}{a(x)} + \frac{a(x)q(x)}{a'(x)}. \tag{2.105}$$

Integrating the above relation, we get

$$\log\left(\frac{a(x)D_0}{a'(x)}\right) = \log a(x) + \int_h^x \frac{a(\xi)q(\xi)}{a'(\xi)}\,d\xi,$$

which implies that

$$\frac{a(x)D_0}{a'(x)} = a(x)e^{\int_h^x \frac{a(\xi)q(\xi)}{a'(\xi)}\,d\xi},$$

i.e.,

$$a(x)b'(x) - b(x)a'(x) = -a'(x)e^{\int_h^x \frac{a(\xi)q(\xi)}{a'(\xi)}\,d\xi}.$$

Dividing by $a^2(x)$, we get

$$\left(\frac{b(x)}{a(x)}\right)' = -\frac{a'(x)}{a^2(x)}e^{\int_h^x \frac{a(\xi)q(\xi)}{a'(\xi)}\,d\xi},$$

so that finally

$$b(x) = -a(x)\int_h^x \frac{a'(\xi)}{a^2(\xi)}e^{\int_h^\xi \frac{a(\eta)q(\eta)}{a'(\eta)}\,d\eta}\,d\xi.$$

This expression when substituted into the expression for $p(x)$ yields

$$p(x) = \frac{a(x)q(x)}{a'(x)} - \frac{q'(x)}{q(x)} + \frac{a''(x)}{a'(x)}, \tag{2.106}$$

and the complementary solutions obtained from Eqns. (2.104) are

$$y_1(x) = e^{-\int_h^x \frac{a(\eta)q(\eta)}{a'(\eta)}\,d\eta} = \frac{a'(x)}{q(x)}e^{-\int_h^x p(\eta)\,d\eta}, \tag{2.107a}$$

$$y_2(x) = e^{-\int_h^x \frac{a(\eta)q(\eta)}{a'(\eta)}\,d\eta}\int_h^x \frac{q(\xi)e^{\int_h^\xi \frac{a(\eta)q(\eta)}{a'(\eta)}\,d\eta}}{a'(\xi)}\,d\xi = \frac{a'(x)e^{-\int_h^x p(\eta)\,d\eta}}{q(x)}\int_h^x \frac{q^2(\xi)e^{\int_h^\xi p(\eta)\,d\eta}}{[a'(\xi)]^2}\,d\xi. \tag{2.107b}$$

Ideally speaking, one would like to solve for $a(x)$ using Eqn. (2.106). As expected, this is more difficult than the original problem. Thus, one specifies $a(x)$ which yields a constraint between $p(x)$ and $q(x)$ given by Eqn. (2.106), and the complementary solutions given by Eqns. (2.107) subject to this constraint.

For various choice of $a(x)$, we have the following:

- If $a(x) = e^{\alpha x^\beta}$, where α and β are constants, then the constraint is

$$p(x) = \frac{x^{1-\beta}q(x)}{\alpha\beta} - \frac{q'(x)}{q(x)} + \frac{\alpha\beta x^\beta + \beta - 1}{x},$$

  and the complementary solutions are

$$y_1(x) = e^{-\int_h^x \frac{\eta^{1-\beta}q(\eta)}{\alpha\beta}\,d\eta},$$
$$y_2(x) = e^{-\int_h^x \frac{\eta^{1-\beta}q(\eta)}{\alpha\beta}\,d\eta}\int_h^x \xi^{1-\beta}e^{-\alpha\xi^\beta}q(\xi)e^{\int_h^\xi \frac{\eta^{1-\beta}q(\eta)}{\alpha\beta}\,d\eta}\,d\xi.$$

- If $a(x) = x^\alpha$, then the constraint is

$$p(x) = \frac{xq(x)}{\alpha} - \frac{q'(x)}{q(x)} + \frac{\alpha - 1}{x}, \tag{2.108}$$

and the complementary solutions are

$$y_1(x) = e^{-\int_h^x \frac{\eta q(\eta)}{\alpha}\,d\eta}, \tag{2.109a}$$

$$y_2(x) = e^{-\int_h^x \frac{\eta q(\eta)}{\alpha}\,d\eta} \int_h^x \xi^{1-\alpha} q(\xi) e^{\int_h^\xi \frac{\eta q(\eta)}{\alpha}\,d\eta}\,d\xi. \tag{2.109b}$$

As an example, consider again Eqn. (2.98), namely,

$$y''(x) + xy'(x) + y(x) = 0,$$

we see that Eqn. (2.108) is satisfied with $\alpha = 1$. Thus, the complementary solutions obtained using Eqns. (2.109) are

$$y_1 = e^{-x^2/2},$$

$$y_2 = e^{-x^2/2} \int_0^x e^{\xi^2/2}\,d\xi,$$

which agree with the solutions given by Eqns. (2.99).

An another example, consider the differential equation

$$y''(x) - \frac{(2x-1)}{x} y'(x) + \frac{(x^2 - x - 1)}{x^2} y(x) = 0.$$

The coefficients $p(x)$ and $q(x)$ do not satisfy the constraint equations given by Eqn. (2.108). However, if we convert the above equation to the following standard form

$$z''(x) - \frac{3}{4x^2} z(x) = 0, \tag{2.110a}$$

$$y = \frac{ze^x}{\sqrt{x}}. \tag{2.110b}$$

then Eqn. (2.110a) satisfies the constraint given by Eqn. (2.108) with $\alpha = -3/2$ and $\alpha = 1/2$. Thus, the complementary solutions obtained using Eqn. (2.109a) are $z_1(x) = 1/\sqrt{x}$ and $z_2(x) = x\sqrt{x}$, which results in $y_1(x) = e^x/x$ and $y_2(x) = xe^x$.

- If $a(x) = \int_h^x e^{\alpha\xi^\beta} q(\xi)\,d\xi$, the constraint equation is

$$q(x) = p'(x) + \alpha\beta x^{\beta-1} p(x) - \alpha\beta x^{\beta-2}\left[\alpha\beta x^\beta + \beta - 1\right].$$

and the complementary solutions are

$$y_1(x) = e^{\alpha x^\beta} e^{-\int_h^x p(\eta)\,d\eta},$$
$$y_2(x) = e^{\alpha x^\beta} e^{-\int_h^x p(\eta)\,d\eta} \int_h^x e^{-2\alpha\xi^\beta} e^{\int_h^\xi p(\eta)\,d\eta}\,d\xi.$$

- If $a(x) = e^{\int_h^x \frac{q(\xi)}{\alpha \log \xi}\,d\xi}$, then y_1 is of the form x^β, a case that we have already considered (see Eqns. (2.94)).
- If $a(x) = e^{\alpha \int_h^x \frac{q(\xi)}{\xi^\beta}\,d\xi}$, then y_1 is of the form $e^{\gamma x^\delta}$, which is again a case that we have already considered (see Eqns. (2.97)).
- If $a(x) = \int_h^x \frac{q(\xi)}{\xi^\beta}\,d\xi$, the constraint is

$$q(x) = p'(x) - \frac{\beta(1+\beta+xp(x))}{x^2},$$

and the complementary solutions are

$$y_1(x) = \frac{1}{x^\beta} e^{-\int_h^x p(\eta)\,d\eta},$$
$$y_2(x) = \frac{e^{-\int_h^x p(\eta)\,d\eta}}{x^\beta} \int_h^x \xi^{2\beta} e^{\int_h^\xi p(\eta)\,d\eta}\,d\xi.$$

As an example, the complementary solutions of

$$y''(x) + (\log x)y'(x) + \frac{y(x)}{x} = 0,$$

are

$$y_1(x) = \frac{e^x}{x^x},$$
$$y_2(x) = \frac{e^x}{x^x} \int_1^x \frac{\xi^\xi}{e^\xi}\,d\xi,$$

while those of

$$y''(x) + \sin x y'(x) + \cos x y(x) = 0,$$

are

$$y_1(x) = e^{\cos x},$$
$$y_2(x) = e^{\cos x} \int_0^x e^{-\cos\xi}\,d\xi.$$

An another example, the complementary solutions of

$$y''(x) + cxy'(x) + c(1-\beta) - \frac{\beta(1+\beta)}{x^2} = 0,$$

are

$$\begin{aligned} y_1(x) &= \frac{1}{x^\beta} e^{-cx^2/2}, \\ y_2(x) &= \frac{1}{x^\beta} e^{-cx^2/2} \int_0^x \xi^{2\beta} e^{c\xi^2/2}\, d\xi. \end{aligned}$$

- If $a(x) = e^{\int_h^x \frac{p(\xi)q(\xi)}{\alpha}\, d\xi}$, the constraint is

$$q(x) = \frac{\alpha\left[p^2(x) - p'(x) - \alpha\right]}{p^2(x)}, \tag{2.111}$$

and the complementary solutions are

$$y_1(x) = e^{-\alpha \int_h^x \frac{1}{p(\eta)}\, d\eta}, \tag{2.112a}$$

$$y_2(x) = e^{-\alpha \int_h^x \frac{1}{p(\eta)}\, d\eta} \int_h^x e^{\int_h^\xi \frac{(2\alpha - p^2(\eta))}{p(\eta)}\, d\eta}\, d\xi. \tag{2.112b}$$

- If $a(x) = e^{\int_h^x \frac{q(\xi)}{\alpha p(\xi)}\, d\xi}$, the constraint is

$$q(x) = \alpha\left[(1-\alpha)p^2(x) + p'(x)\right], \tag{2.113}$$

and the complementary solutions are

$$y_1(x) = e^{-\alpha \int_h^x p(\eta)\, d\eta}, \tag{2.114a}$$

$$y_2(x) = e^{-\alpha \int_h^x p(\eta)\, d\eta} \int_h^x e^{\int_h^\xi (2\alpha-1)p(\eta)\, d\eta}\, d\xi. \tag{2.114b}$$

Note that if the constraint given by Eqn. (2.113) is satisfied for $\alpha = 1/2$, then $v(x) = 0$ in the standard form (see Eqn. (2.83b)). As an example, the differential equation given by Eqn. (2.91) satisfies the constraint given by Eqn. (2.113) with $\alpha = 1/2$, and thus, the solution given by Eqns. (2.114) agrees with that given by Eqns. (2.92).

- If $a(x) = \int_h^x p(\xi)q(\xi)\, d\xi$, the constraint is

$$q(x) = \frac{2p(x)p'(x) - p''(x)}{p(x)}, \tag{2.115}$$

and the complementary solutions are

$$y_1(x) = p(x)e^{-\int_h^x p(\eta)\,d\eta}, \tag{2.116a}$$

$$y_2(x) = p(x)e^{-\int_h^x p(\eta)\,d\eta} \int_h^x \frac{1}{p^2(\xi)} e^{\int_h^\xi p(\eta)\,d\eta}\,d\xi. \tag{2.116b}$$

- If $a(x) = \int_h^x \frac{q(\xi)}{p(\xi)}\,d\xi$, the constraint is

$$q(x) = \frac{p(x)p''(x) - 2[p'(x)]^2}{p^2(x)}, \tag{2.117}$$

and the complementary solutions are

$$y_1(x) = \frac{1}{p(x)} e^{-\int_h^x p(\eta)\,d\eta}, \tag{2.118a}$$

$$y_2(x) = \frac{1}{p(x)} e^{-\int_h^x p(\eta)\,d\eta} \int_h^x p^2(\xi) e^{\int_h^\xi p(\eta)\,d\eta}\,d\xi. \tag{2.118b}$$

2.5.8 Series solutions for second order variable coefficient equations

We shall just illustrate this method by an example. The Bessel functions of the first and second kind, denoted by $J_n(r)$ and $Y_n(r)$ are two linearly independent solutions of the differential equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) + \left(1 - \frac{n^2}{r^2}\right)u = 0, \tag{2.119}$$

where n is a nonnegative integer (Actually, one can define Bessel functions for even complex-valued n, but we shall restrict ourselves to integers) . Thus, the general solution of the above differential equation can be written as $u = c_1 J_n(r) + c_2 Y_n(r)$. The Bessel function of the first kind for a nonnegative integer n is defined in terms of an infinite series as

$$J_n(x) := \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x^2}{4}\right)^k.$$

The Bessel function of the second kind $Y_n(x)$ can also be expressed as a series, but the expression is more complicated. The Bessel function $Y_n(x)$ is singular at $r = 0$, so that c_2 is set to zero in solutions for domains that include the origin. However, on domains such as hollow cylinders, both functions need to be included in the general solution.

From Eqn. (2.63), we get $W(x) = W(x_0)e^{-\log x} = W(x_0)/x$. Taking $W(1) = 2/\pi$, we get $W(x) = 2/(\pi x)$. Thus, if $y_1 := J_n(x)$ and $y_2 := Y_n(x)$, then, by the expression for the Wronskian, we get

$$y_2' y_1 - y_2 y_1' = \frac{2}{\pi x}.$$

By multiplying by the integrating factor $1/y_1^2$, we get

$$\left(\frac{y_2}{y_1}\right)' = \frac{2}{\pi x y_1^2},$$

which finally yields

$$\frac{y_2(x)}{y_1(x)} - \frac{y_2(a)}{y_1(a)} = \int_a^x \frac{2}{\pi \xi y_1^2(\xi)}\, d\xi,$$

or

$$\frac{Y_n(x)}{J_n(x)} - \frac{Y_n(a)}{J_n(a)} = \int_a^x \frac{2}{\pi \xi J_n^2(\xi)}\, d\xi.$$

For arbitrary λ_m and λ_n, and with H denoting either $J_n(x)$ or $Y_n(x)$, we have

$$\begin{aligned}\int r H_\nu(\lambda_m r) H_\nu(\lambda_n r)\, dr &= \frac{r\left[\lambda_n H_{\nu-1}(\lambda_n r) H_\nu(\lambda_m r) - \lambda_m H_{\nu-1}(\lambda_m r) H_\nu(\lambda_n r)\right]}{\lambda_m^2 - \lambda_n^2}, \quad \lambda_m \neq \lambda_n, \\ &= \frac{r^2}{2}\left[H_\nu^2(\lambda_n r) - H_{\nu-1}(\lambda_n r) H_{\nu+1}(\lambda_n r)\right], \quad \lambda_m = \lambda_n.\end{aligned}$$

Thus, if λ_m, $m = 1, 2, \ldots, \infty$, denote the roots of J_ν, and δ_{mn} denotes the Kronecker delta, then the Bessel functions J_ν are orthogonal in the following sense:

$$\int_0^R r J_\nu\left(\lambda_m \frac{r}{R}\right) J_\nu\left(\lambda_n \frac{r}{R}\right) dr = \frac{R^2}{2}\left[J_{\nu+1}(\lambda_m)\right]^2 \delta_{mn}. \tag{2.120}$$

The Legendre equation is given by

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dH}{d\xi}\right] + \left[\nu(\nu+1) - \frac{m^2}{1-\xi^2}\right] H = 0, \tag{2.121}$$

The solution is given by

$$H(\xi) = C_\nu P_\nu^m(\xi) + D_\nu Q_\nu^m(\xi),$$

where P^m and Q^m are the associated Legendre functions of the first and second kind, respectively. The solutions $P_\nu^m(\xi)$ diverges for $\xi = -1$ unless ν is a non-negative integer, which we now denote by n. For $m = 0$, Eqn. (2.121) reduces to

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dH}{d\xi}\right] + n(n+1)H = 0. \tag{2.122}$$

One set of solutions of the above equation, known as the Legendre polynomials, is given by the Rodrigues formula as

$$P_n(\xi) = \frac{1}{2^n n!}\frac{d^n}{d\xi^n}(\xi^2 - 1)^n. \tag{2.123}$$

Thus, again, we see that the solution is obtained in the form of a series in ξ. For domains that include the axis in axisymmetric problems, the functions $Q_n(\xi)$ can be excluded since they are singular.

From Eqn. (2.63), we get

$$W(x) = \frac{W(x_0)}{x^2 - 1}.$$

Taking $W(0) = 1$, we get $W(x) = 1/(1 - x^2)$, so that with $y_1(x) := P_n(x)$ and $y_2(x) := Q_n(x)$, we have

$$y_2' y_1 - y_2 y_1' = \frac{1}{1 - x^2}.$$

As in the Bessel equation case, this leads to

$$\frac{y_2(x)}{y_1(x)} = \int \frac{dx}{(1 - x^2) y_1^2(x)},$$

or

$$Q_n(x) = P_n(x) \int \frac{dx}{(1 - x^2) P_n^2(x)}. \tag{2.124}$$

The associated Legendre polynomials satisfies the following orthogonality relation:

$$\int_{-1}^{1} P_k^{(m)}(\xi) P_l^{(m)}(\xi)\, d\xi = \frac{2}{2l + 1}\frac{(l + m)!}{(l - m)!}\delta_{kl},$$

which for $m = 0$ reduces to

$$\int_{-1}^{1} P_k(\xi) P_l(\xi)\, d\xi = \frac{2}{2l + 1}\delta_{kl}.$$