Indian Institute of Science, Bangalore

ME 257: Midsemester Test

Date: 24/2/2018.
Duration: 9.30 a.m.–11.00 a.m.
Maximum Marks: 100

1. Carry out the following steps : (60)

   (a) Using indicial notation, find α, β, γ and δ in the following identities:

   $$\boldsymbol{\nabla} \cdot [\boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b})] = \alpha(\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}) + \beta \boldsymbol{a} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{b})], \tag{1a}$$
   $$\boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla} \cdot \boldsymbol{b})\boldsymbol{a}] = \gamma(\boldsymbol{\nabla} \cdot \boldsymbol{a})(\boldsymbol{\nabla} \cdot \boldsymbol{b}) + \delta \boldsymbol{a} \cdot [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{b})], \tag{1b}$$

   where $\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{b}$ is given by

   $$w_i = \epsilon_{ijk} \frac{\partial b_k}{\partial x_j}.$$

   (b) On the domain Ω with boundary Γ, consider the functional given by

   $$\Pi = \frac{1}{2} \int_\Omega \left[(\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) + (\boldsymbol{\nabla} \cdot \boldsymbol{u})^2 \right] d\Omega - \int_\Gamma \bar{\boldsymbol{h}} \cdot \boldsymbol{u} \, d\Gamma, \tag{2}$$

   where $\bar{\boldsymbol{h}}$ is a given quantity. By taking the first variation, derive the variational formulation, and subsequently using Eqns. (1), the strong form of the governing equation, and the boundary condition on Γ. You may directly use the relation $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$. *Justify all steps in your derivation.*

2. The region between two concentric circles of inner and outer radii a and b, respectively, has a displacement field of the form $u_r = 0$ and $u_\theta = u_\theta(r)$, where r is the radial coordinate in a polar coordinate system (ignore z, u_z etc. in this problem). The region is subjected to the essential boundary condition $u_\theta(a) = 0$, and a natural boundary condition (to be specified later) at $r = b$. (40)

   (a) Derive the variational formulation corresponding to the following governing equation for $\tau_{r\theta}(r)$ (Hint: Combine terms under one derivative):

   $$\frac{d\tau_{r\theta}}{dr} + \frac{2\tau_{r\theta}}{r} = 0. \tag{3}$$

   (b) What is the allowable natural boundary condition at $r = b$? Impose a nonzero natural boundary condition with value T_0 at $r = b$, and incorporate the essential boundary condition at $r = a$, and the natural boundary condition at $r = b$ into your variational formulation.

(c) Substitute the following constitutive relation into your variational formulation:

$$\tau_{r\theta} = \mu r \frac{d}{dr}\left(\frac{u_\theta}{r}\right),$$

where μ is the shear modulus. The bilinear form $a(u_\theta, v)$ associated with the resulting variational formulation should be symmetric.

(d) Using one linear element (two nodes), formulate the stiffness matrix (which should be symmetric since $a(u_\theta, v)$ is symmetric) in terms of the strain-displacement matrix $\boldsymbol{B}$ (give the expression for this matrix in natural coordinates), and the load vector. *Do not* carry out the integrations with respect to the natural coordinate ξ.

(e) Incorporate the essential boundary condition with K_{11}, K_{12} etc. denoting the entries of the stiffness matrix (since you have not carried out the integrations).

(f) Find the exact solution to this problem using the governing equation

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(ru_\theta)}{dr}\right) = 0.$$