Indian Institute of Science

ME 261: Final Exam

Date: 3/12/18.
Duration: 9.00 a.m.–12.00 noon.
Maximum Marks: 100

Solve $(1, 2, 3)$, $(4, 5, 6)$, and $(7, 8, 9)$ as three 'bunches' so that it is easier for us to grade.

1. Using the relation $\mathbf{cof}\,\boldsymbol{T} = (\det \boldsymbol{T})\boldsymbol{T}^{-T}$ for an invertible $\boldsymbol{T}$ (14)

   (a) Find the eigenvalues of $\mathbf{cof}\,\boldsymbol{T}$ in terms of the eigenvalues of $\boldsymbol{T}$ (denoted by λ_1, λ_2 and λ_3). (Hint: First find the eigenvalues of $(\mathbf{cof}\,\boldsymbol{T})^T$, and then find the relation between the eigenvalues of a tensor and those of its transpose.)

   (b) Using the above result, find the principal invariants of $e^{\mathbf{cof}\,\boldsymbol{T}}$ in terms of $(\lambda_1, \lambda_2, \lambda_3)$. Also find a relation between the third principal invariant of $e^{\mathbf{cof}\,\boldsymbol{T}}$ and the second principal invariant of $\boldsymbol{T}$.

   (c) If $\boldsymbol{T} \equiv \boldsymbol{S} \in$ Sym, and if $\boldsymbol{S} = \sum_{n=1}^{3} \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*$, then using the above results, find the spectral resolution of $e^{\mathbf{cof}\,\boldsymbol{S}}$. Find the polar decomposition factors $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ of $-e^{-\mathbf{cof}\,\boldsymbol{S}}$. Guesswork is allowed provided you show that the factors satisfy all the required properties.

   Justify all steps in your derivation.

2. Let (8)

$$\boldsymbol{x} = \boldsymbol{Q}(t)\boldsymbol{X} + \boldsymbol{c}(t),$$

   where $\boldsymbol{X}$ and $\boldsymbol{c}(t)$ are vectors, and $\boldsymbol{Q}(t) \in \text{Orth}^+$. Let $\boldsymbol{\omega}(t)$ be the axial vector of $\dot{\boldsymbol{Q}}\boldsymbol{Q}^T \in$ Skw, where a dot indicates a derivative with respect to time.

   (a) Find expressions for

$$\boldsymbol{v}(\boldsymbol{x}, t) = \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{X}},$$
$$\boldsymbol{a}(\boldsymbol{x}, t) = \left(\frac{\partial^2 \boldsymbol{x}}{\partial t^2}\right)_{\boldsymbol{X}},$$

   in terms of $\boldsymbol{\omega}$, $\dot{\boldsymbol{\omega}}$, $\boldsymbol{c}$, $\dot{\boldsymbol{c}}$, $\ddot{\boldsymbol{c}}$ and $\boldsymbol{x}$.

   (b) Using the above result find expressions for $\boldsymbol{v}(\boldsymbol{x}_2, t) - \boldsymbol{v}(\boldsymbol{x}_1, t)$ and $\boldsymbol{a}(\boldsymbol{x}_2, t) - \boldsymbol{a}(\boldsymbol{x}_1, t)$ in terms of $\boldsymbol{x}_2 - \boldsymbol{x}_1$.

   (c) Given $\boldsymbol{v}(\boldsymbol{x}_1, t) = \boldsymbol{a}(\boldsymbol{x}_1, t) = \boldsymbol{c}(t) = \boldsymbol{0}$, $\boldsymbol{x}_2 - \boldsymbol{x}_1 = R\boldsymbol{e}_r$, $\boldsymbol{\omega} = \dot{\theta}(t)\boldsymbol{e}_z$, where R is a constant and $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ represent cylindrical basis vectors, find expressions for $\boldsymbol{v}(\boldsymbol{x}_2, t)$ and $\boldsymbol{a}(\boldsymbol{x}_2, t)$ with respect to $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ using the results of part (b) above.

3. Let $\boldsymbol{\omega} := \boldsymbol{\nabla} \times \boldsymbol{u}$. (18)

   (a) Find a relation between $\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u})$, $(\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{u}$ and $\boldsymbol{u} \times \boldsymbol{\omega}$.

   (b) Use this relation to find a relation between

$$\boldsymbol{\nabla} \times \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{u}\right), \quad \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{\nabla}\boldsymbol{\omega})\boldsymbol{u}, \ (\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{\omega} \text{ and } (\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{\omega}.$$

   Assume that the spatial and temporal derivatives can be interchanged. *Justify* all steps in your derivation.

4. Solve the initial value problem (10)

$$t^2 \frac{dx}{dt} - x^2 = xt, \qquad x(1) = 1.$$

5. Consider the following inhomogeneous second order ODE: (20)

$$(2t+1)(t+1)\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - 2x = 4t^2 + 1 + 4t. \tag{1}$$

   (a) Verify that $x_1 = t$ is a solution to the corresponding homogeneous equation (RHS $= 0$ in Eqn. 1).

   (b) Using this solution, find another linearly independent solution x_2 of the homogeneous equation. Show explicitly that x_1 and x_2 are linearly independent.

   (c) Using x_1 and the x_2 you have determined, find the general solution to the inhomogeneous problem given by Eqn. 1.

6. Hermite's equation (of order 2) is given by: (10)

$$\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 4x = 0. \tag{2}$$

   (a) Show that $x_1 = 1 - 2t^2$ is a solution to Hermite's equation

   (b) Find a second linearly independent solution x_2

   (c) Verify explicitly that x_1 and x_2 are linearly independent. Write down the general solution to Eqn. 2.

7. For the complex function (5)

$$f(z) = \frac{\pi \sin(\pi z)}{z^3},$$

   find the residue at $z = 0$.

8. Using a suitable contour evaluate the following integral: (10)

$$I = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{(x+2)(x+3)} dx.$$

   Show all the steps clearly and state all the theorems used. Show the rigorous steps where needed.

9. Given the Bromwich integral (5)

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds, \quad \text{find} \quad f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right).$$