Indian Institute of Science

ME 261: Final Exam

Date: 3/12/15.
Duration: 2.00 p.m.–5.00 p.m.
Maximum Marks: 100

Solve $(1,2,3,4)$, $(5,6,7)$ and $(8,9)$ as three 'bunches' so that it is easier for us to grade.

1. Let $\boldsymbol{Q} \in \text{Orth}^+$. If (10)

$$(\operatorname{tr} \boldsymbol{Q})^2 - \operatorname{tr}(\boldsymbol{Q}^2) = \alpha(\operatorname{tr} \boldsymbol{Q}),$$

determine α.

2. Let $\boldsymbol{R}$ and $\boldsymbol{S}$ be two second-order tensors. By using (5)

$$I_2(\boldsymbol{T}) = \frac{1}{2}\left[(\operatorname{tr} \boldsymbol{T})^2 - \operatorname{tr}(\boldsymbol{T}^2)\right],$$

determine if

$$I_2(\boldsymbol{RS}) = I_2(\boldsymbol{SR}).$$

Prove any results that you use along the way (even if they appear in the notes). If the result is not true, provide a counterexample.

3. Let $\boldsymbol{S} \in \text{Sym}$. If $\boldsymbol{S}$ is positive definite, is $e^{\boldsymbol{S}}$ positive definite? Is the converse also true? If any of the results is not true, provide a counterexample. (5)

4. Let $\boldsymbol{v}(\boldsymbol{x}, t)$ be a vector-valued field, and let (20)

$$\boldsymbol{D} = \frac{1}{2}\left[\boldsymbol{\nabla}_x \boldsymbol{v} + (\boldsymbol{\nabla}_x \boldsymbol{v})^T\right],$$
$$\boldsymbol{\tau} = -p(\boldsymbol{x}, t)\boldsymbol{I} + \lambda(\operatorname{tr} \boldsymbol{D})\boldsymbol{I} + 2\mu\boldsymbol{D},$$

where λ and μ are constants, and p is a function of $(\boldsymbol{x}, t)$.

(a) Let ρ be a constant. Substitute the above two equations into the right-hand-side of

$$\rho\left[\left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_x + (\boldsymbol{\nabla}_x \boldsymbol{v})\boldsymbol{v}\right] = \boldsymbol{\nabla}_x \cdot \boldsymbol{\tau}$$

so as to obtain a tensorial equation for $\boldsymbol{v}$.

(b) Let $\boldsymbol{v} = \boldsymbol{\nabla}_x \phi$, where ϕ is a harmonic function, i.e., $\boldsymbol{\nabla}_x^2 \phi = 0$. Substitute $\boldsymbol{v} = \boldsymbol{\nabla}_x \phi$ into the equation for $\boldsymbol{v}$ that you have derived in part (a) above, and find an equation for $p(\boldsymbol{x}, t)$ in terms of ϕ (Hint: Is p dependent on $|\boldsymbol{\nabla}_x \phi|$).

5. Find the general solution of $x^2y' = y^2+xy-x^2$ given that $y = x$ is a particular solution. (10)

6. Verify that $y = x$ and $y = 1/(x+1)$ are linearly independent solutions of (10)

$$(2x+1)\,(x+1)\,y'' + 2xy' - 2y = 0.$$

Hence solve

$$(2x+1)\,(x+1)\,y'' + 2xy' - 2y = (2x+1)^2.$$

7. Obtain the general solution of (20)

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 3x + z + e^{2t}\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -x + 4y + z + e^{2t}\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= 4x - 4y + 2z + t - e^{2t}.\end{aligned}$$

8. Use contour integration (on a suitable contour) and find the value of the following integral (show all the steps clearly): (10)

$$I = \int_0^\infty \frac{x^{1/2}}{1+\sqrt{2}\,x+x^2}dx.$$

9. Use contour integration (on a suitable contour) and find the value of the following integral (show all the steps clearly): (10)

$$I = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}dx.$$

For problem (2) include two poles inside the closed contour.

Some relevant formulae

$$(\mathbf{cof}\,\boldsymbol{T})^T\boldsymbol{T} = (\det \boldsymbol{T})\boldsymbol{I}.$$

If $f(z)$ is a ratio of two polynomials $p(z)$ and $q(z)$ then the residue at $z = z_o$ is also found as

$$\mathrm{Res}_{z=z_o}\frac{p(z)}{q(z)} = \frac{p(z_o)}{q'(z_o)}$$

provided $q'(z_o) \neq 0$. If $q(z_o) = 0$, then this is the only way.