Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Note: Some relevant formulae are given at the end which you can directly use. Derive any other formulae that you may require.

Date: 25/4/2019.
Duration: 9.00 a.m.–12.00 noon
Maximum Marks: 100

1. The equation for the deflection w of a membrane is given by (20)

$$\boldsymbol{\nabla}^2 w + q = 0,$$

where q is a distributed load acting on the membrane.

(a) Assuming $w = 0$ on the boundary, develop the variational formulation corresponding to the above governing equation.

(b) Using $\boldsymbol{\nabla} w = (\frac{\partial w}{\partial r}, \frac{1}{r}\frac{\partial w}{\partial \theta})$, and using appropriate symmetry considerations, specialize your variational formulation to the case of a circular membrane of radius a with $w|_{r=a} = 0$ and $q = q_0$, where q_0 a constant.

(c) Using a one-parameter Rayleigh-Ritz approximation for w, where your approximating function should satisfy appropriate conditions *both* at $r = 0$ and at $r = a$, find an approximate solution for w.

2. The governing partial differential equation for a vector-valued field variable $\boldsymbol{h}(\boldsymbol{x}, t)$ is given by (35)

$$\mu \frac{\partial \boldsymbol{h}}{\partial t} + \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times \boldsymbol{h} - \boldsymbol{a} \times \boldsymbol{h}] = \boldsymbol{0}, \quad \in \Omega, \tag{1}$$

where $\boldsymbol{a}(\boldsymbol{x})$ is a given vector which is zero on the boundary Γ of the domain Ω.

(a) Develop the variational formulation corresponding to Eqn. (1) under the boundary conditions (i) $\boldsymbol{h}$ specified on Γ_h and (ii) $(\boldsymbol{\nabla} \times \boldsymbol{h}) \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}$ where $\boldsymbol{g}$ is a given vector on the remaining part of the boundary Γ_q.

(b) Consider the axisymmetric case where $\boldsymbol{h} = (h_r(r, z), 0, h_z(r, z))$, $\boldsymbol{a} = (a_r(r, z), 0, a_z(r, z))$ and $\boldsymbol{g} = g(r, z, t)\boldsymbol{e}_\theta$ so that

$$\boldsymbol{\nabla} \times \boldsymbol{h} = \left(\frac{\partial h_r}{\partial z} - \frac{\partial h_z}{\partial r} \right) \boldsymbol{e}_\theta.$$

Find the '$\boldsymbol{B}$' matrix that links $\boldsymbol{\nabla} \times \boldsymbol{h}$ to the nodal degrees of freedom of a 3-node triangular element. This $\boldsymbol{B}$ matrix should be expressed in terms of N_i and/or its derivatives $\partial N_i/\partial \xi$ and $\partial N_i/\partial \eta$, $i = 1, 2, 3$.

Express $d\Omega$ in terms of $d\xi\, d\eta$, and finally express the '$\boldsymbol{K}^{(e)}$' (which could be unsymmetric) and $\boldsymbol{M}^{(e)}$ matrices for a single element, and the load vector in terms of $\boldsymbol{B}$, $\boldsymbol{N}$, $\boldsymbol{A}$ (a matrix containing the components a_r and a_z), $\boldsymbol{g}$ and $\boldsymbol{n}$ without carrying out the integrations but with the proper integration limits stated. You can state the load vector in terms of $\boldsymbol{g}$, $\boldsymbol{n}$ and $d\Gamma$ without simplifying since you will be asked to do that in the next part.

(c) If $\boldsymbol{g} = \boldsymbol{e}_\theta$ on the edge of a 9-node quadrilateral whose nodes in the local numbering scheme are say $(2, 6, 3)$, with their coordinates in the (r, z) plane given by $\boldsymbol{x}_2 = (1, -1)$, $\boldsymbol{x}_6 = (2, 0)$ and $\boldsymbol{x}_3 = (1, 1)$, find the consistent load vector corresponding to the degrees of freedom on this edge (*don't* state the zero loads corresponding to nodes other than nodes 2, 6 and 3).

(d) Develop a time-stepping strategy for the semi-discrete axisymmetric formulation assuming that the element matrices that you developed in part (b) above have been assembled into global matrices which you now denote by $\boldsymbol{M}$, $\boldsymbol{K}$ etc., and state the appropriate initial conditions.

3. A disc of inner radius a, outer radius b, and unit width along the z-direction is fixed rigidly at the inner boundary $r = a$, and on the outer boundary $r = b$ is subjected to tangential tractions $s_0(t)$ which are independent of θ but dependent on time t (see Fig. 1). The radial displacement is given by (45)

$$u_r = c_1 \cos\theta + c_2 \sin\theta + \frac{c_3}{r},$$

where c_1, c_2 and c_3 are constants to be determined. Assuming $u_z = 0$ and treating the problem as two-dimensional (i.e., $u_r = u_r(r, \theta, t)$ and $u_\theta = u_\theta(r, \theta, t)$) and using appropriate symmetry conditions, reduce the following governing equations

$$\rho \frac{\partial^2 u_r}{\partial t^2} = (\lambda + \mu)\frac{\partial(\operatorname{tr}\boldsymbol{\epsilon})}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r u_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + -\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right],$$

$$\rho \frac{\partial^2 u_\theta}{\partial t^2} = (\lambda + \mu)\frac{1}{r}\frac{\partial(\operatorname{tr}\boldsymbol{\epsilon})}{\partial \theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right] + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + +\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right],$$

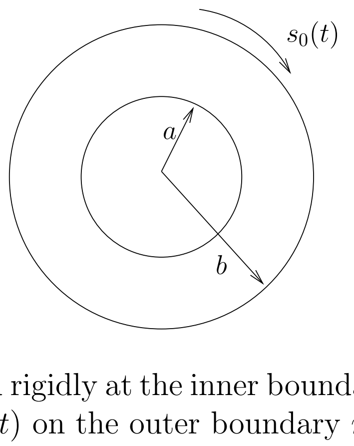

Figure 1: Hollow disk fixed rigidly at the inner boundary $r = a$, and subjected to tangential tractions $s_0(t)$ on the outer boundary $r = b$.

where

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \epsilon_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right],$$

$$\epsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \qquad \operatorname{tr}\boldsymbol{\epsilon} = \epsilon_{rr} + \epsilon_{\theta\theta}.$$

to a single partial differential equation for $u_\theta(r, \theta, t)$.

(a) Using the partial differential equation for u_θ that you have found, develop the variational formulation over the two-dimensional region bounded by the circles $r = a$ and $r = b$. Use the constitutive relation $\boldsymbol{\tau} = \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}$ to simplify the boundary terms in your variational formulation.

(b) By discretizing with respect to the space variables, develop the semi-discrete formulation in terms of the mass and stiffness matrices and the load vector. For a single linear element, derive the strain-displacement matrix $\boldsymbol{B}$ *pertaining to your variational formulation* using natural coordinates, and state the mass and stiffness matrices in terms of $\boldsymbol{N}$ and $\boldsymbol{B}$. *Do not evaluate the integrals in these expressions.* For two linear elements (3 nodes), evaluate the consistent global load vector (any unknown reactions can be denoted simply by R without evaluation).

(c) By choosing the variation of u_θ to be $(r, \partial u_\theta/\partial t)$ in your variational formulation, derive the two quantities (let us call them 'momentum' and 'energy') that are conserved in the continuum setting if $s_0 = 0$, and the boundary at $r = a$ is also suddenly made traction free. If $s_0 = 0$, but the boundary at $r = a$ is still kept fixed rigidly, is momentum and/or energy conserved? Justify your answers.

(d) Develop a time-stepping strategy for your semi-discrete formulation (with appropriate initial conditions that you must state) that is 'energy-momentum' conserving in the fully discrete setup. Prove that your time-stepping strategy conserves these two quantities. In these proofs, *justify* why the choices that you make for the variations are admissible.

Some Relevant Formulae

For a quadratic 1-D element:

$$N_1 = -\frac{\eta(1-\eta)}{2}, \quad N_2 = 1 - \eta^2, \quad N_2 = \frac{\eta(1+\eta)}{2}.$$