Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 10/12/18.
Duration: 9.00 a.m.–12.00 noon
Maximum Marks: 100

Instructions:

You may directly use the formulae at the back.

1. In what follows λ and μ are constants, $\tilde{\boldsymbol{v}}$ is the Lagrangian velocity vector, a superposed dot denotes a material derivative, $\boldsymbol{V}$ and $\boldsymbol{R}$ are the polar decomposition factors of $\boldsymbol{F}$, $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$, $\boldsymbol{S}$ is the second Piola-Kirchhoff stress, $\boldsymbol{\tau}$ is the Cauchy stress, $\boldsymbol{q}$ is the heat flux, $\boldsymbol{g}$ is the temperature gradient, $\boldsymbol{F}^* = \boldsymbol{Q}(t)\boldsymbol{F}$ and $\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}$. Find expressions for $\boldsymbol{L}^*$, $\boldsymbol{D}^*$, $\boldsymbol{W}^*$, $\boldsymbol{R}^*$, $\boldsymbol{V}^*$ and $\boldsymbol{g}^*$, where $\boldsymbol{L} = \boldsymbol{\nabla}_x\boldsymbol{v}$, $\boldsymbol{D} = (\boldsymbol{L} + \boldsymbol{L}^T)/2$, $\boldsymbol{W} = (\boldsymbol{L} - \boldsymbol{L}^T)/2$ and $\boldsymbol{g} = \boldsymbol{\nabla}_x\theta$. (30)

   (a) Determine if the following constitutive relations are frame-indifferent:

   $$\boldsymbol{S} = \mu\boldsymbol{\nabla}_X\tilde{\boldsymbol{v}},$$
   $$\boldsymbol{S} = \lambda\dot{J}\boldsymbol{C}^{-1} + \mu\left[(\mathbf{cof}\,\boldsymbol{F})^T\dot{\boldsymbol{F}}\boldsymbol{C}^{-1} + \boldsymbol{C}^{-1}\dot{\boldsymbol{F}}^T\mathbf{cof}\,\boldsymbol{F}\right],$$
   $$\boldsymbol{\tau} = \mu(\boldsymbol{R}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{R}^T).$$

   (b) Find a relation between $\boldsymbol{w} := \boldsymbol{\nabla}_x \times \boldsymbol{v}$ and $\boldsymbol{W}$. Use this relation to find a relation between $\boldsymbol{w}^*$ and $\boldsymbol{w}$. Take $\boldsymbol{\Omega}$ to be the axial vector of $\dot{\boldsymbol{Q}}\boldsymbol{Q}^T$. Use this relation to find if the following constitutive equations are frame indifferent:

   $$\boldsymbol{\tau} = \lambda(\boldsymbol{\nabla}\cdot\boldsymbol{w})\boldsymbol{I} + \mu[(\boldsymbol{\nabla}\boldsymbol{w}) + (\boldsymbol{\nabla}\boldsymbol{w})^T],$$
   $$\boldsymbol{q} = (\boldsymbol{\nabla}\boldsymbol{w})\boldsymbol{g},$$
   $$\dot{\boldsymbol{q}} - \boldsymbol{W}\boldsymbol{q} = \mu\boldsymbol{B}\boldsymbol{g}.$$

2. Consider the torsion of of an incompressible circular cylinder of initial radius and length R_0 and L, respectively, which is fixed at the bottom $z = 0$ (see Fig. 1). Normal and tangential tractions are applied at the top surface $z = L$ so that the length remains L at all times, and generate a torque $T(t)\boldsymbol{e}_z$ which is assumed to be given. The constitutive relation is given by $\boldsymbol{\tau} = -p\boldsymbol{I} + \mu\boldsymbol{B}$ where the pressure p is independent of θ, and μ is a constant. Assume the deformation to be given by (30)

   $$r = aR,$$
   $$\theta = \Theta + f(Z,t),$$
   $$z = Z,$$

   where a is a constant, and assume that a body force $\rho\boldsymbol{b} = [\rho r(\omega_0^2 - (\partial f/\partial t)^2) - \mu r(\partial f/\partial Z)^2]\,\boldsymbol{e}_r$, where ω_0 is a given constant, acts on the cylinder. The lateral surface of the cylinder is traction free.

(a) Find a and the governing differential equation for $f(Z,t)$. Do not attempt to solve this equation. Assuming that the cylinder is initially in a state of rest, state the initial and boundary conditions on $f(Z,t)$ in terms of known quantities such as $T(t)$. (Hint: Exercise care while finding the governing equation for $f(Z,t)$ since $(\boldsymbol{e}_r, \boldsymbol{e}_\theta)$ change with time.)

(b) Find the normal force exerted on the top surface $Z = L$ of the cylinder.

3. Consider the setup shown in Fig. 2 where two masses of mass M rotate about the $\boldsymbol{e}_3$ axis. The masses M are connected to a mass m and the ground by means of massless, rigid links of length L; all joints are pin-joints. The $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ frame of reference rotates with a given angular velocity $\boldsymbol{\Omega} = \omega(t)\boldsymbol{e}_3$ with respect to a fixed frame of reference in which the body force is $-g\boldsymbol{e}_3$. A normal force $F\boldsymbol{e}_2$ (indicated by 'x') is exerted on the masses M so as to keep them in the $\boldsymbol{e}_1$-$\boldsymbol{e}_3$ plane. The mass m is connected to the ground by means of a spring with spring constant k. At $t = 0$ the system is in a state of rest with $\theta(0) = \theta_0$, and the spring is undeformed in this position. Assuming the masses to be concentrated at their center of mass, find the governing equation for $\theta(t)$ along with the appropriate initial conditions, and the force F on any one of the masses M. Do not attempt to solve the equation for $\theta(t)$. You may directly take $\boldsymbol{Q}^T\boldsymbol{b}^* = -g\boldsymbol{e}_3$ in the body force transformation formula since gravity is the only body force in the fixed frame of reference. *Justify* all assumptions that you make. (40)

Some relevant formulae

$$\mathbf{cof}\,(\boldsymbol{AB}) = (\mathbf{cof}\,\boldsymbol{A})(\mathbf{cof}\,(\boldsymbol{B}),$$

$$\det(\boldsymbol{T}+\boldsymbol{U}) = \det\boldsymbol{T} + \mathbf{cof}\,\boldsymbol{T} : \boldsymbol{U} + \mathbf{cof}\,\boldsymbol{U} : \boldsymbol{T} + \det\boldsymbol{U},$$

$$w_i = -\frac{1}{2}\epsilon_{ijk}W_{jk},$$

$$W_{ij} = -\epsilon_{ijk}w_k,$$

$$\boldsymbol{b} = \boldsymbol{Q}^T\left[\boldsymbol{b}^* - \ddot{\boldsymbol{c}}\right] - \dot{\boldsymbol{\Omega}} \times \boldsymbol{x} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) - 2\boldsymbol{\Omega} \times \boldsymbol{v}.$$

$$F_{iJ} = \frac{h_i}{h_J}\frac{\partial\hat{\chi}_i}{\partial\eta_J}, \quad \text{no sum on } i,\ J, \quad h_i \equiv (1, r, 1), \quad h_J \equiv (1, R, 1).$$

If $\boldsymbol{\tau}$ is symmetric tensor-valued field, then the components of $\boldsymbol{\nabla}_x \cdot \boldsymbol{\tau}$ with respect to a cylindrical coordinate system are

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_r = \frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\tau_{rr}-\tau_{\theta\theta}}{r},$$

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_\theta = \frac{\partial\tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r},$$

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_z = \frac{\partial\tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{z\theta}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r}.$$

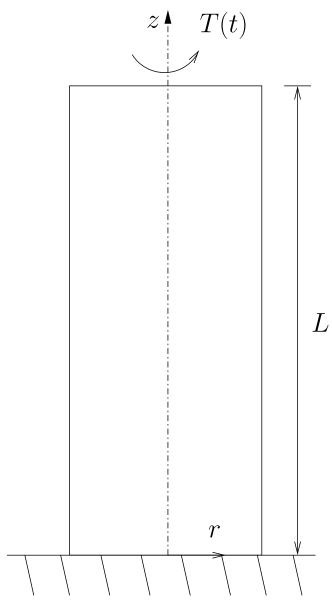

Figure 1: A circular cylinder of initial radius and length R_0 and L, respectively, subjected to a torque $T(t)$ on its top surface.

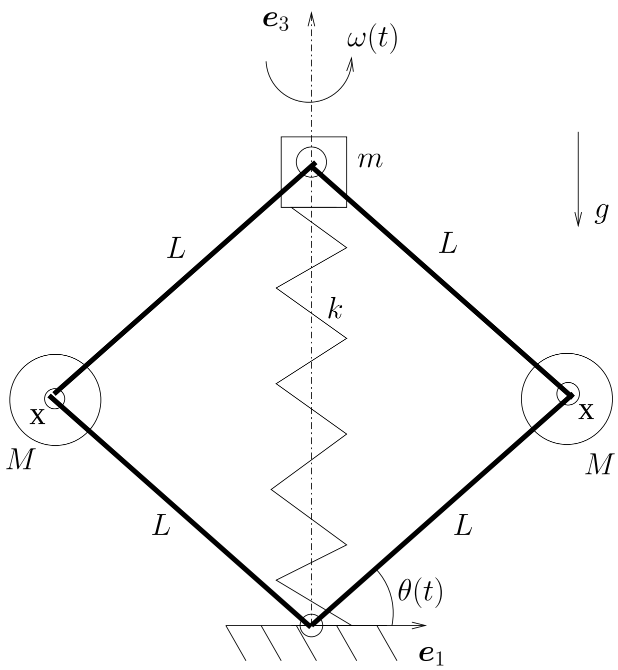

Figure 2: Problem 3.